Bi-Periodic $k$-Pell Sequence

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Abstract

In this paper, we define the bi-periodic $k$-Pell sequence. We obtain Binet’s formula, some identities of the bi-periodic $k$-Pell sequences like Catalan, Cassini and D’Ocagne’s identities as well as some related summation formulas.

1 Introduction

For any natural number $n$ and non-zero real numbers $a$ and $b$, bi-periodic Fibonacci sequence was defined recursively by Edson and Yayenie [1] as

$$q_n = \begin{cases} \ aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ b q_{n-1} + q_{n-2}, & \text{if } n \text{ is odd,} \end{cases}, \quad n \geq 2$$

with initial condition $q_0 = 0, q_1 = 1$. Bi-periodic Lucas sequence was defined recursively by Bilgici [2] as

$$l_n = \begin{cases} \ a l_{n-1} + l_{n-2}, & \text{if } n \text{ is even} \\ b l_{n-1} + l_{n-2}, & \text{if } n \text{ is odd,} \end{cases}, \quad n \geq 2$$

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with initial condition \( l_0 = 2, l_1 = a \). In [3], Uygun and Owusu defined bi-periodic Jacobsthal sequence as

\[
J_0 = 0, \quad J_1 = 1, \quad J_n = \begin{cases} 
aJ_{n-1} + 2J_{n-2}, & \text{if } n \text{ is even,} 
bJ_{n-1} + 2J_{n-2}, & \text{if } n \text{ is odd,}
\end{cases} \quad n \geq 2.
\]

Uygun and Karatas [4] defined bi-periodic Pell-Lucas sequence as

\[
Q_0 = 2, \quad Q_1 = 2a, \quad Q_n = \begin{cases} 
2aQ_{n-1} + Q_{n-2}, & \text{if } n \text{ is odd,} 
2bQ_{n-1} + Q_{n-2}, & \text{if } n \text{ is even,}
\end{cases} \quad n \geq 2.
\]

In [1, 2, 3, 4], the Binet’s formula, identities such as Catalan, Cassini and the D’Ocagne’s as well as some related summation formulas were also given. In this paper, we define bi-periodic \( k \)-Pell sequence and some identities are also given.

### 2 Main results

**Definition 2.1.** Let \( k \) be a positive real number. For any two non-zero real numbers \( a \) and \( b \) where \( ab \) does not contain any values belonging to the interval \([-k, 0]\), the bi-periodic \( k \)-Pell sequence, denoted by \( \{P_{k,n}\}_{n=0}^{\infty} \), is defined recursively by

\[
P_{k,0} = 0, \quad P_{k,1} = 1, \quad P_{k,n} = \begin{cases} 
2aP_{k,n-1} + kP_{k,n-2}, & \text{if } n \text{ is odd,} 
2bP_{k,n-1} + kP_{k,n-2}, & \text{if } n \text{ is even,}
\end{cases} \quad n \geq 2.
\]

The first five elements of the bi-periodic \( k \)-Pell sequence are

\[
P_{k,0} = 0, \quad P_{k,1} = 1, \quad P_{k,2} = 2b, \quad P_{k,3} = 4ab + k, \quad P_{k,4} = 4b(2ab + k).
\]

When \( a = b = k = 1 \), we have a Pell sequence. If we set \( a = b = 1 \), for some positive real number \( k \), we get a \( k \)-Pell sequence. If \( k = 1 \), for some non-zero real number \( a \) and \( b \), we get a bi-periodic Pell sequence. From the above definition, we have the nonlinear quadratic equation for bi-periodic \( k \)-Pell sequence by

\[
x^2 - 4abx - 4abk = 0 \quad (2.1)
\]

with roots \( \alpha \) and \( \beta \) defined by

\[
\alpha = 2 \left( ab + \sqrt{ab(ab + k)} \right), \quad \beta = 2 \left( ab - \sqrt{ab(ab + k)} \right).
\]
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Note that $\alpha \beta = -4abk, (\alpha + k)(\beta + k) = k^2, ab(\alpha + k) = \frac{\alpha^2}{4}, ab(\beta + k) = \frac{\beta^2}{4}, \alpha(\beta + k) = -k\beta$.

**Lemma 2.2.** For any positive real number $k$, the bi-periodic $k$-Pell sequence satisfies the following properties

(i) $P_{k, 2n} = (4ab + 2k) P_{k, 2n-2} - k^2 P_{k, 2n-4}$,

(ii) $P_{k, 2n+1} = (4ab + 2k) P_{k, 2n-1} - k^2 P_{k, 2n-3}$.

**Proof.** By using definition 2.1,

(i) $P_{k, 2n} = 4abx P_{k, 2n} - 2x + 2bk P_{k, 2n} - 3x + k P_{k, 2n} - 5x = (4ab + 2x) P_{k, 2n} - 2x - k^2 P_{k, 2n} - 4$,

(ii) $P_{k, 2n+1} = 4abx P_{k, 2n+1} + 2ak P_{k, 2n+2} + k P_{k, 2n+1} = (4ab + 2x) P_{k, 2n+1} - k^2 P_{k, 2n+3}$.

**Theorem 2.3.** The generating function for the bi-periodic $k$-Pell sequence is given by

$$P(x) = \frac{x(1 + 2bx - kx^2)}{1 - (4ab + 2k)x^2 + k^2 x^4}. \quad (2.2)$$

**Proof.** The generating function $P(x) = \sum_{m=0}^{\infty} P_{k, 2m} x^{2m} + \sum_{m=0}^{\infty} P_{k, 2m+1} x^{2m+1}$ is divided into two parts (even and odd). The even and odd parts of the series are denoted by $P_0(x)$ and $P_1(x)$, respectively. We have $P_0(x) = 2bx^2 + \sum_{m=2}^{\infty} P_{k, 2m} x^{2m}$ and $P_1(x) = x + (4ab + k)x^3 + \sum_{m=2}^{\infty} P_{k, 2m+1} x^{2m+1}$. Then

$$P_0(x) = \frac{2bx^2}{1 - (4ab + 2k)x^2 + k^2 x^4} \text{ and } P_1(x) = \frac{x(1 + 2bx - kx^2)}{1 - (4ab + 2k)x^2 + k^2 x^4}.$$

Thus $P(x) = \frac{x(1 + 2bx - kx^2)}{1 - (4ab + 2k)x^2 + k^2 x^4}$.

**Theorem 2.4.** (Binet’s Formula) For any positive real number $k$ and natural number $n$, the bi-periodic $k$-Pell sequence is given by the following

$$P_{k, n} = \frac{(2b)^{1-\xi(n)}}{4ab} \left[ \frac{\alpha^n - \beta^n}{\alpha - \beta} \right], \quad (2.3)$$

where the parity function

$$\xi(n) = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$$

and

$$\xi(n) = n - 2 \left\lfloor \frac{n}{2} \right\rfloor.$$
Proof. The generating function $P(x)$ can be expressed using partial fraction decomposition as:

$$
P(x) = \frac{1}{k^2} \left[ \frac{\alpha k x}{\beta - \alpha} - \frac{\alpha^2}{2a(\beta - \alpha)} x^2 - \frac{\alpha k}{\beta - \alpha} \right] + \frac{\beta k x}{\beta - \alpha} + \frac{\beta^2}{2a(\beta - \alpha)} x^2.
$$

The Maclaurin series expression of the function $\frac{Ax + B}{x^2 - C}$ is expressed in the form

$$
\frac{Ax + B}{x^2 - C} = - \sum_{n=0}^{\infty} AC^{-n-1} x^{2n+1} - \sum_{n=0}^{\infty} BC^{-n-1} x^{2n},
$$

the generating function $P(x)$ can be expanded as:

$$
P(x) = \frac{1}{k^2} \left[ - \sum_{n=0}^{\infty} \left( \frac{\alpha k}{\beta - \alpha} \right) \left( \frac{\alpha + k}{k^2} \right)^{n-1} x^{2n+1} - \sum_{n=0}^{\infty} \left( \frac{-\alpha^2}{2a(\beta - \alpha)} \right) \left( \frac{\alpha + k}{k^2} \right)^{n-1} x^{2n} 
- \sum_{n=0}^{\infty} \left( \frac{-\beta k}{\beta - \alpha} \right) \left( \frac{\beta + k}{k^2} \right)^{n-1} x^{2n+1} - \sum_{n=0}^{\infty} \left( \frac{\beta^2}{2a(\beta - \alpha)} \right) \left( \frac{\beta + k}{k^2} \right)^{n-1} x^{2n} \right].
$$

By $(\alpha + k)(\beta + k) = k^2, ab(\alpha + k) = \frac{\alpha^2}{4}, ab(\beta + k) = \frac{\beta^2}{4}, \alpha(\beta + k) = -k\beta$ and the definition of $\xi(n)$, we have

$$
P(x) = \frac{1}{k^2} \left[ \frac{k^2}{\beta - \alpha} \sum_{n=0}^{\infty} \left( \frac{\beta^{2n+1} - \alpha^{2n+1}}{(4ab)^n} \right) x^{2n+1} + \frac{2bk^2}{\beta - \alpha} \sum_{n=0}^{\infty} \left( \frac{\beta^{2n} - \alpha^{2n}}{(4ab)^n} \right) x^{2n} \right]
= \sum_{n=0}^{\infty} \frac{(2b)^{1-\xi(n)}}{(4ab)^{n}} \frac{(\alpha^n - \beta^n)}{\alpha - \beta} x^n.
$$

Since $P(x) = \sum_{n=0}^{\infty} P_{k,n} x^n = \sum_{n=0}^{\infty} \left( \frac{(2b)^{1-\xi(n)}}{(4ab)^{n}} \frac{(\alpha^n - \beta^n)}{\alpha - \beta} \right) x^n$, then

$$
P_{k,n} = \frac{(2b)^{1-\xi(n)}}{(4ab)^{n}} \frac{(\alpha^n - \beta^n)}{\alpha - \beta}.
$$
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**Theorem 2.5.** The limit of every two consecutive terms of bi-periodic $k$-Pell sequences are

$$
\lim_{n \to \infty} \frac{P_{k,2n+1}}{P_{k,2n}} = \frac{\alpha}{2b} \quad \text{and} \quad \lim_{n \to \infty} \frac{P_{k,2n}}{P_{k,2n-1}} = \frac{\alpha}{2a}.
$$

(2.4)

**Proof.** By Binet’s formula and \( \lim_{n \to \infty} \left( \frac{\beta}{\alpha} \right)^n = 0 \), we have

$$
\lim_{n \to \infty} \frac{P_{k,2n+1}}{P_{k,2n}} = \lim_{n \to \infty} \frac{(2b)^{1-\xi(2n+1)}}{(4ab)^{\left\lfloor \frac{2n+1}{2} \right\rfloor} \left( \frac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta} \right)} = \frac{1}{2b} \lim_{n \to \infty} \frac{1 - \left( \frac{\beta}{\alpha} \right)^{2n+1}}{1 - \left( \frac{\beta}{\alpha} \right)^n} = \frac{\alpha}{2b}.
$$

$$
\lim_{n \to \infty} \frac{P_{k,2n}}{P_{k,2n-1}} = \lim_{n \to \infty} \frac{(2b)^{1-\xi(2n)}}{(4ab)^{\left\lfloor \frac{2n}{2} \right\rfloor} \left( \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} \right)} = \frac{1}{2a} \lim_{n \to \infty} \frac{1 - \left( \frac{\beta}{\alpha} \right)^{2n}}{1 - \left( \frac{\beta}{\alpha} \right)^n} = \frac{\alpha}{2a}.
$$

\( \square \)

**Theorem 2.6.** For any positive real number $k$ and positive integer $n$,

$$
P_{k,-n} = \frac{(-1)^{n-1}}{k^n} P_{k,n}.
$$

(2.5)

**Proof.** According to Theorem 2.4 and $\alpha \beta = -4abk$, we have

$$
P_{k,-n} = \frac{(2b)^{1-\xi(-n)}}{(4ab)^{\left\lfloor \frac{n}{2} \right\rfloor} \left( \frac{\alpha^{-n} - \beta^{-n}}{\alpha - \beta} \right)}
$$

$$
= \frac{(2b)^{1-\xi(n)}}{(4ab)^{\left\lfloor \frac{n}{2} \right\rfloor} (\alpha - \beta)} \left( \frac{(-1)(\alpha^n - \beta^n)}{(-4abk)^n} \right)
$$

$$
= \frac{(-1)^{n-1}}{k^n} \left( \frac{(2b)^{1-\xi(n)}}{(4ab)^{\left\lfloor \frac{n}{2} \right\rfloor} \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right)} \right)
$$

$$
= \frac{(-1)^{n-1}}{k^n} P_{k,n}.
$$

\( \square \)
Theorem 2.7. (Catalan’s Identity) For any integers \( n \) and \( r \) with \( n \geq r \), we have
\[
\frac{(2b)^{2(r+1)}}{(4ab)^{n-r-\xi(r)}} P_{k,n-r} P_{k,n+r} - \frac{(2b)^{2(\xi(r+1)-(1)^{(n)}\xi(r))}}{(4ab)^{\xi(n)-\xi(r)}} P_{k,n}^2 = (-1)^{n-r+1} k^{n-r} (2b)^{2(\xi(n)+1)-(1)^{(n)}\xi(r))} P_{k,n}^2.
\]
Proof. Since
\[
\frac{(2b)^{2(r+1)}}{(4ab)^{n-r-\xi(r)}} P_{k,n-r} P_{k,n+r} = \frac{(2b)^{2(\xi(n)+1)-(1)^{(n)}\xi(n)+\xi(r+1))}}{(4ab)^{n-\xi(r)(\alpha-\beta)^2}} (\alpha^{2n} - \alpha^{n-r} \beta^{n+r} - \alpha^{n+r} \beta^{n-r} + \beta^{2n})
\]
and
\[
\frac{(2b)^{2(\xi(r+1)-(1)^{(n)}\xi(r))}}{(4ab)^{\xi(n)-\xi(r)}} P_{k,n}^2 = \frac{(2b)^{2(\xi(n)+1)+\xi(r+1)-(1)^{(n)}\xi(r))}}{(4ab)^{n-\xi(r)(\alpha-\beta)^2}} (\alpha^{2n} - 2\alpha^{n} \beta^{n} + \beta^{2n}),
\]
we have
\[
\frac{(2b)^{2(r+1)}}{(4ab)^{n-r-\xi(r)}} P_{k,n-r} P_{k,n+r} - \frac{(2b)^{2(\xi(r+1)-(1)^{(n)}\xi(r))}}{(4ab)^{\xi(n)-\xi(r)}} P_{k,n}^2 = (-1)^{n-r+1} k^{n-r} (2b)^{2(\xi(n)+1)-(1)^{(n)}\xi(r))} P_{k,n}^2.
\]
\( \square \)

Corollary 2.8. (Cassini’s Identity) For any positive real number \( k \) and positive integer \( n \),
\[
\frac{1}{(4ab)^{\xi(n-1)-1}} P_{k,n-1} P_{k,n+1} - \frac{(2b)^{-2(\xi(n))}}{(4ab)^{\xi(n)-1}} P_{k,n}^2 = (-1)^{n} k^{n-1} (2b)^{2(\xi(n+1)-(1)^{(n)}\xi(n))}.
\]
Proof. For \( r = 1 \) in Catalan’s identity, we have Cassini’s identity. \( \square \)

Theorem 2.9. (D’Ocagne’s Identity) For any positive real number \( k \) and positive integers \( m, n \) with \( m \geq n \),
\[
\frac{(2b)^{\xi(m+2)+\xi(n)+\xi(m-n+1)}}{(4ab)^{\frac{n+\xi(m)-\xi(n)-\xi(m-n+1)}{2}}} P_{k,m} P_{k,n+1} - \frac{(2b)^{\xi(m+1)+\xi(n+2)+\xi(m-n+1)}}{(4ab)^{\frac{n-\xi(m)+\xi(n)-\xi(m-n)}{2}}} P_{k,m+1} P_{k,n}
\]
\[
= 4(-k)^{(n)} b^2 (4ab)^{\frac{n}{2}} P_{k,m-n}.
\]

Proof. By Binet’s formula, \( \left[ \frac{n+1}{2} \right] = \frac{n + \xi(n)}{2} \) and \( 1 - \xi(n) = \xi(n+1) \), and we have
\[
\frac{(2b)^{\xi(m+2)+\xi(n+1)+\xi(m-n)}}{(4ab)^{\frac{m+\xi(m)-\xi(n)-\xi(m-n)}{2}}} P_{k,m} P_{k,n+1} - \frac{(2b)^{\xi(m+1)+\xi(n+2)+\xi(m-n)}}{(4ab)^{\frac{m-\xi(m)+\xi(n)-\xi(m-n)}{2}}} P_{k,m+1} P_{k,n}
\]
\[
= \frac{(2b)^{2+\xi(m-n+1)}}{(4ab)^{\frac{m-\xi(m-n)}{2}}(\alpha-\beta)^2} (\alpha^{m+n+1} - \alpha^{m} \beta^{n+1} - \alpha^{n+1} \beta^{m} + \beta^{m+n+1})
\]
\[
- \frac{(2b)^{2+\xi(m-n+1)}}{(4ab)^{\frac{m-\xi(m-n)}{2}}(\alpha-\beta)^2} (\alpha^{m+n+1} - \alpha^{m+1} \beta^{n} - \alpha^{n} \beta^{m+1} + \beta^{m+n+1})
\]
Theorem 2.10. For any positive real number \( k \) and non-negative integer \( n \), we have

(i) \[ \sum_{r=0}^{n} \binom{n}{r} (2b)^{\xi(r)} (4ab)^{\frac{m-n}{2}} k^{n-r} \bigg( \frac{2b}{4ab} \bigg)^{\xi(r)+\frac{m-n}{2}} P_{k,r} = P_{k,2n}, \]

(ii) \[ \sum_{r=0}^{n} \binom{n}{r} (2b)^{\xi(r+1)} (4ab)^{\xi(r)+\frac{m-n}{2}} k^{n-r} P_{k,r+1} = 2b P_{k,2n+1}. \]

Proof. To prove (i) and (ii), we use Theorem 2.4.

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