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Root elements and root subgroups in $E_6(K)$ for fields K of characteristic two

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Abstract

The purpose of this article is to give an elementary description of the root elements and the root subgroups of the Chevalley group E of type $E_6(K)$ in fields K of characteristic two. We show that there is a bijection between the root subgroups of E_6 and the family V_6 of all 6-dimensional submodules of the 27-dimensional module E_6 over K. Then we give a construction of the stabilizer of a 6-dimensional Tits subspace in V_6 which is the maximal parabolic subgroup P_6 in $E_6(K)$.

1 Introduction

Subgroups of the group $E_6(q)$, $q = p^a$, $p \geq 5$, which are generalized by root-subgroups were considered by Cooperstein [13]. In [3], a brief description of the groups $E_6(K)$, ${}^2E_6(K)$ and $F_4(K)$ was given, where the root involutions and root subgroups were defined. For more information about Lie algebras of type E_6 and their adjoint Chevalley groups, one may refer to [1,3,4,5,7,8,11,12].

It is remarkable to mention that most of the available literature on Lie algebras and Chevalley groups does not deal with fields of characteristic two.

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Hence this study provides a new interpretation of root system and root subgroups. Such interpretations are useful since they may lead to new insights and more efficient ways of computation concerning finite simple groups of Lie type.

2 Notations and general setup

Consider a 6-dimensional vector space V over the Galois field $\mathbb{F}_{\not{2}}$ endowed with a non-degenerate quadratic form of minimal Witt-index. Let $(v \mid \omega) = Q(v+\omega)+Q(v)+Q(\omega)$ be the associated bilinear form to Q on V. Let $\mathbb{B} = \{0 \neq x \in V \mid Q(x) = 0\}$ be the set of points and $\mathcal{L} = \{L < V \mid \dim L = 2$ and $Q(L) = 0\}$ be the set of lines and the set of non-singular vectors s of V are the exterior points. The order of $\mathbb{B}$ is 27 and the order of $\mathcal{L}$ is 45 . The pair $(\mathbb{B}, \mathcal{L})$ is the generalized quadrangle of type $\mathrm{O}_6^-(2)$ and the Weyl group $W = \{g \in GL(V) \mid Q(x^g) = Q(x) \forall x \in V\}$. This group is a 3-transposition group generated by 36 reflections σ_s, s is an exterior point and $v^{\sigma_s} = v+(v|s)s$ for $v \in V$. For these observatations see [2].

Remark 2.1. *If Δ is a root base, then $s = s_\Delta = \sum_{x \in \Delta} x$ is an exterior point, and $\Delta^* = s_\Delta + \Delta$ is also a root base. We call Δ and Δ^* corresponding root bases.*

Moreover, we denote by Δ_0 the set of all points which are orthogonal to s_Δ, so that $\mathbb{B} = \Delta \cup \Delta^* \cup \Delta_0$.

Definition 2.1. Let K be a field of characteristic 2 and let A be a vector space over K with basis $\{e_x \mid x \in B\}$.

Definition 2.2. For a root base Δ and $k \in K$, define the root-elements (Chevalley generators) $r_\Delta(k) \in GL(A)$ by

$$e_x^{r_\Delta(k)} = \begin{cases} e_x + k e_x^{\sigma_\Delta} & , \quad x \in \Delta \\ e_x & , \quad \text{otherwise} \end{cases}$$

Definition 2.3. The commutator $[A, r_\Delta(k)]$ is defined by,

$$[A, r_\Delta(k)] = \langle e_x + e_x^{r_\Delta(k)} \mid x \in \mathbb{B} \rangle.$$

It is clear that $[A, r_\Delta(k)] = \langle e_y \mid y \in \Delta^* \rangle$ is of dimension 6, if $k \neq 0$.

Definition 2.4. For a root base Δ, the corresponding root-subgroup $U_\Delta = U_\Delta(K) = \{r_\Delta(k) \mid k \in K\}$. The group generated by all root-subgroups is denoted by $\mathbb{E}(K)$ or simply $\mathbb{E}$.

Definition 2.5. Define a quadratic map $\mathbb{Q}$ from A into A by $\hat{Q}(a) = \sum_{x\in\mathbb{B}} Q_x(a)e_x$, where Q_x is the quadratic form on A defined as

$$Q_x(a) = \sum_{\{x,y,z\}\in\mathfrak{L}} a_y a_z, \; with \; a = \sum_{x\in\mathbb{B}} a_x e_x$$

Lemma 2.1. [3] *The group $\mathbb{E} = \mathbb{E}(K)$ module its center is isomorphic to the Chevalley group $E_6(K)$ or simply E_6.*

Remark 2.2. *The 27-dimensional vector space A over K with basis $\{e_x \mid x \in \Omega\}$ can be turned into a commutative, non-associative algebra. For $x, y \in \Omega$, set*

$$e_x e_y = \begin{cases} e_{x+y} & , \; x \neq y \; and \; (x|y) = 0 \\ 0 & , \; otherwise. \end{cases}$$

Definition 2.6. Define the inner product $\langle|\rangle$ on A by: $\langle e_i|e_j\rangle = \begin{cases} 1 & , \; i = j \\ 0 & , \; otherwise \end{cases}$ and define a symmetric trilinear form T on A by

$$T(e_x, e_y, e_z) = \begin{cases} 1 & , \; \{x, y, z\} \in \mathfrak{L} \\ 0 & , \; \text{otherwise} \end{cases}$$

In Particular, $T(a, b, c) = \langle ab|c\rangle = \langle a|bc\rangle$ for all $a, b, c \in A$.

Proposition 2.1. [10]

1. *$\hat{Q}(ka) = k^2\hat{Q}(a)$ for $k \in K$ and $a \in A$.*
2. *$\hat{Q}(e_x) = 0$ for points x.*
3. *$ab = \hat{Q}(a + b) - \hat{Q}(a) - \hat{Q}(b)$ for $a, b \in A$.*

Remark 2.3. [10] *If $G = \{g \in GL(A) \mid T(a^g, b^g, c^g) = T(a, b, c) \; \forall a, b, c \in A\}$, then*

$$G = \left\{g \in GL(A) \mid a^g b^g = (ab)^{g^*} \; \forall a, b, c \in A\right\},$$

where g^* is the transposed inverse of g with respect to basis e_x , $x \in \mathbb{B}$.

Definition 2.7. Let $G_0 = \{g \in GL(A) \mid \hat{Q}(a^g) = \hat{Q}(a)^{g^*} \; \forall a \in A\}$.

Theorem 2.1. [10] $\mathbb{E} \leq G_0 \leq G$.

3 Root elements and root subgroups

Now, we introduce a new definition for root subgroups of $\mathbb{E}$ and we show that the two notions are equivalent.

Definition 3.1. Let $a, b \in A$ where $a = \sum_{x \in \mathbb{B}} a_x e_x$, $b = \sum_{x \in \mathbb{B}} b_x e_x$. Then the inner product $\langle a|b \rangle$ of a and b is defined as $\langle a|b \rangle = \sum_{x \in \mathbb{B}} a_x b_x$.

Definition 3.2. Let $a, b \in A$ with $\hat{Q}(a) = \hat{Q}(b) = 0$ and $\langle a|b \rangle = 0$. For such a, b, define $t_{a,b}$ by $z^{t_{a,b}} = z + (za)b + \langle z|b \rangle a$ for $z \in A$.

Definition 3.3. Let V_6 be the family of all singular subspaces of A of dimension 6; i.e., $V_6 = \Big\{ U \leq A \mid \dim(U) = 6 \text{ and } \hat{Q}(U) = 0 \Big\}$. We define the root-subgroups in the following way:
For $U \in V_6$, define $R_U = \{ g \in \langle \, \rangle \mid [A, g] \leq U \leq C_A(g) \}$.

Proposition 3.1. *Let Δ be a root base and $U = \langle e_x \mid x \in \Delta \rangle$ and let $t \in \mathbb{E}$ such that $[A, t] = U < C_A(t)$. Then $t = r_{\Delta^*}(k)$ for some $0 \neq k \in K$ and any $r_{\Delta^*}(k)$, $k \neq 0$, has this property.*

Proof. For all $x \in \Delta$ and for all $y \in \mathbb{B}$, $e_x^t = e_x$ *and* $e_y^t = e_y + u_y$ for some $u_y \in U$ as $[A, t] = U$. Hence if $y \in \Delta^*$, then $e_y^t = e_y + u_y$ and $e_y u_y = \hat{Q}(e_y + u_y) = \hat{Q}(e_y^t) = \hat{Q}(e_y)^{t^*} = 0^{t^*} = 0$, using Proposition 2.1 and Theorem 2.1.
Let $u_y = \sum_{z \in \Delta} k_z e_z$, then $0 = e_y u_y = \sum_{z} k_z e_y e_z = \sum_{z \neq y^\sigma} k_z e_{y+z}$, and hence, $u_y = k_y e_{y^\sigma}$, where σ is the reflection corresponding to s_Δ. Also, if $y_1, y_2 \in \Delta^*$, $y_1 \neq y_2$, then $(e_{y_1} + k_{y_1} e_{y_1^\sigma})(e_{y_2} + k_{y_2} e_{y_2^\sigma}) = e_{y_1}^t e_{y_2}^t = (e_{y_1} e_{y_2})^{t^*} = 0$, which implies $0 = k_{y_1} e_{y_1^\sigma} e_{y_2} + k_{y_2} e_{y_1} e_{y_2^\sigma} = (k_{y_1} + k_{y_2}) e_{y_1 + y_2 + s_\Delta}$, from which it follows that $k_{y_1} = k_{y_2}$. Hence, $e_x^t = e_x$ for all $x \in \Delta$, and $e_y^t = e_y + k e_{y^\sigma}$ for $y \in \Delta^*$. It remains to discuss the case $z \in \Delta_0$ which implies $z = x + y + s$ for $x, y \in \Delta$, $x \neq y$. For $y_1 \in \Delta \setminus \{x, y\}$, it follows that $\langle y_1 + s | x + y + s \rangle = \langle y_1 | x + y + s \rangle = 1$ which implies $e_{y_1+s} \cdot e_z = 0$. As $e_z^t = e_z + u_z$ and $e_{y_1+s}^t = e_{y_1+s} + k e_{y_1}$ as $y_1 + s \in \Delta^*$, we have

$$
\begin{aligned}
(e_z + u_z)(e_{y_1+s} + k e_{y_1}) &= e_z^t e_{y_1+s}^t = (e_z e_{y_1+s})^{t^*} = 0 \\
&= k e_z e_{y_1} + u_z e_{y_1+s} = u_z e_{y_1+s} \ \text{ as } \langle y_1 | z \rangle = \langle y_1 + s | z \rangle = 1.
\end{aligned}
$$

If $u_z = \sum_{x \in \Delta} k_x e_x$, then $0 = e_{y_1+s} u_z = \sum_{x \in \Delta} k_x e_{y_1+s} e_x = \sum_{x \neq y_1} k_x e_{y_1+x+s}$ which implies $k_x = 0$ for all $x \neq y_1$. For $y_1 \neq x, y$, $u_z = k_{y_1} e_{y_1}$. Take another y_1'.

Then $u_z = k_{y_1'} e_{y_1'}$ which implies $u_z = 0$. Hence $t = r_{\Delta^*}(k)$ for some $k \in K$. The converse holds by Definition 2.6. $\square$

Corollary 3.1. *The two definitions of root-subgroups are equivalent.*

Proof. There is a bijection between the root subgroups and V_6; i.e., a map sends the root-subgroup U_Δ to $[A, U_\Delta] \in V_6$. The above proposition shows that

$$\{g \in \langle\ \rangle \mid [A, g] \leq U \leq C_A(g)\} = \{r^*_\Delta(k) \mid k \neq 0\}\,.$$

As $\mathbb{E}$ is transitive on V_6, the claim follows using [9, Theorem 2.1]. $\square$

Lemma 3.1. *Let $g \in \mathbb{E} \leq G_0$. Then $t^g_{a,b} = t_{a^g, b^{g^*}}$.*

Proof. Consider

$$\begin{aligned} z^{g^{-1}(t_{a,b})g} &= \left(z^{g^{-1}} + (z^{g^{-1}}a)b + \langle z^{g^{-1}}|b\rangle a\right)^g = z + ((z^{g^{-1}}a)b)^g + \langle z|b^{g^*}\rangle a^g \\ &= z + (za^g)b^{g^*} + \langle z|b^{g^*}\rangle a^g \text{ as } ((z^{g^{-1}}a)b)^g = (z^{g^{-1}}a)^{g^*} b^{g^*} = (za^g)b^{g^*} \end{aligned}$$

Hence the claim follows. $\square$

Lemma 3.2. *Let $e_p, b \in A$ for $p \in \mathbb{B}$ with $\hat{Q}(e_p) = \hat{Q}(b) = 0$, $\langle e_p|b\rangle = 0$, and $b = b_0 + b_1$ where $b_0 \in A_0(p)$ and $b_1 \in A_1(p)$. Then $\left[A, t_{e_p,b}\right] = \left[A, t_{e_p,b_1}\right] = \langle e_p\rangle + (e_pA)b_1 \in V_6$, where $A_0(p) = \langle e_x \mid (p|x) = 0\rangle$ and $A_1(p) = \langle e_x \mid (p|x) = 1\rangle$.*

Proof. As $\langle e_p|b\rangle = 0$ and as $\hat{Q}(b) = 0$, it follows that $0 = \hat{Q}(b_0 + b_1) = \hat{Q}(b_0) + \hat{Q}(b_1) + b_0b_1$, which implies $\hat{Q}(b_0) = 0$, $\hat{Q}(b_1) = 0$, $b_0b_1 = 0$ as $\hat{Q}(b_0) \in \langle e_p\rangle$, $\hat{Q}(b_1) \in A_0(p)$, $b_0b_1 \in A_1(p)$.
Let $x \in \mathbb{B}$. Then $e_x^{t_{e_p,b}} = e_x + (e_xe_p)b + \langle e_x|b\rangle e_p = e_x + (e_xe_p)b + b_xe_p$ where $b_x = \langle e_x|b\rangle$. Set $t = t_{e_p,b}$. Hence $e^t_p = e_p$. If $(x|p) = 1$, then $e^t_x = e_x + 0 \cdot b + b_xe_p = e_x + b_xe_p$. If $(x|p) = 0$, then $e^t_x = e_x + e_{x+p}(b_0 + b_1) + b_xe_p = e_x + e_{x+p}b_1 + e_{x+p}b_0 + b_xe_p = e_x + e_{x+p}b_1$ as $e_{x+p}b_0 + b_xe_p = 0$. Because $b_0 = \sum_{(y|p)=0} k_ye_y$, we have $e_{x+p}b_0 = \sum_{(y|p)=0} k_ye_{x+p}e_y$ $= k_xe_{x+p}e_x = k_xe_p = b_xe_p$. Hence, $t_{e_p,b} = t_{e_p,b_1}$ and $t_{e_p,b} = I$ if and only if $b_1 = 0$. If $b_1 \neq 0$, then $\left[A, t_{e_p,b_1}\right] = \langle e_p\rangle + (e_pA)b_1 = \langle e_p\rangle + (e_pA)b$, and $\langle e_p\rangle + (e_pA)b_1 \in V_6$ as it is conjugate under Levi(p) to $\langle e_p\rangle + (e_pA)e_w$ for $w \in \mathbb{B}$ with $(p, w) = 1$, where Levi$(p) = \{r_\Delta(k) \mid k \neq 0, (p, s) = 0\}$. Hence the claim obtains. $\square$

Lemma 3.3. *The subspace $[A, t_{a,b}] = \langle a\rangle + (aA)b \in V_6$, where $a, b \in A$, $\hat{Q}(a) = \hat{Q}(b) = \langle a|b\rangle = 0$.*

Proof. As E is transitive on V_6 [9], then there exists $g \in \mathbb{E} \leq G_0$ with $a^g = e_p$ for $p \in \mathbb{B}$ and hence $t^g_{a,b} = t_{a^g,b^{g^*}} = t_{e_p,c}$ where $c = b^{g^*}$. Hence $t_{a,b} = I$ or $t_{a,b} \neq I$ and $[A, t_{a,b}]^g = \left[A, t_{a^g,b^{g^*}}\right] = [A, t_{a^g,c}]$ by Lemma 3.1 and Lemma 3.2. This implies
$[A, t_{a^g,c}] = \langle e_p\rangle + (e_pA)c = \langle a^g\rangle + (a^gA)b^{g^*} = \langle a^g\rangle + ((aA)b)^g$ or $[A, t_{a,b}] = \langle a\rangle + (aA)b \in V_6$. Hence the claim follows. □

Theorem 3.1. *The set* $\left\{t_{a,b} \neq I \mid \hat{Q}(a) = \hat{Q}(b) = 0\ ,\ \langle a|b\rangle = 0\right\}$ *forms a conjugacy class* $r_\Delta(1)^{\mathbb{E}}$ *in* $\mathbb{E}$ *containing all root elements* $r_\Delta(k)\ ,\ k \neq 0$*, and each root element in* $\mathbb{E}$ *can be written as* $t_{a,b}$ *for suitable elements* $a, b \in A$.

Proof. In Lemma 3.2 it has been shown that any such element $t_{a,b} \neq I$ is conjugate to an element t_{e_p,e_w} for $p, w \in \mathbb{B}$ with $(p, w) = 1$, and $t_{e_p,e_w} = \sigma_{p+w}$ is conjugate to $r_\Delta(1)$ for a root base Δ with $s_\Delta = p + w$. In a matrix form,
$\sigma_{s_\Delta} = \begin{bmatrix} I & I & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$ and $\sigma_{p+w} = \begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$.

In other words, $\{t_{a,b} \neq I \mid \hat{Q}(b) = \hat{Q}(a) = \langle a|b\rangle = 0\}$ is a conjugacy class $r_\Delta(1)^{\mathbb{E}}$, in $\mathbb{E}$ containing the root-element $r_\Delta(k)\ ,\ k \neq 0$. Hence the claim rsults. □

Remark 3.1. *K admits an automorphism of order 2, written as* $x \longrightarrow \bar{x}$. *For* $a = \sum_{x\in\mathbb{B}} a_xe_x \in A$, *set* $\bar{a} = \sum_{x\in\mathbb{B}} \bar{a}_xe_x$, *and define the unitary from* $(a|b) = \langle a|\bar{b}\rangle$ *on* A.
Define $U = \{g \in G_0 \mid g$ *preserves the unitary form; i.e.,* $g^* = \bar{g}$ *where* $g^* = (g^t)^{-1}\}$. *Then* $U/Z(U)$ *is the simple group* ${}^2E_6(\bar{K})$. *If* $K = \mathbb{F}_{q^2}$, *then* $\bar{x} = x^q\ ,\ q$ *is a power of a prime.*

For $a \in A$ *with* $\hat{Q}(a) = 0$ *and* $(a|a) = 0 = \langle a|\bar{a}\rangle$, *set* $t_a = t_{a,\bar{a}}$. *Then* t_a *is a root element or* I. *If* $g \in U$, *then* $t^g_a = t_{a^g}$ *as* $t^g_a = (t_{a,\bar{a}})^g = t_{a^g,\bar{a}^{g^*}} = t_{a^g,\bar{a}^{\bar{g}}} = t_{a^g}$, *this implies that* $t_a \in U$ *and the set* $\left\{t_a \neq I \mid \hat{Q}(a) = 0 = (a|a)\right\}$ *are the long root elements in* ${}^2E_6(\bar{K})$.

Definition 3.4. Let X be a coclique in $\mathbb{B}$ and $A(x) = \langle e_x|x \in X\rangle$. Let $N_{\mathbb{E}}(A(X))$
be the stabilizer of $A(X)$ in $\mathbb{E}$ and $H(A(X)) = \langle r_\Delta(k) \mid r_\Delta(k) \in N_{\mathbb{E}}(A(X))\rangle$ $= \langle U_\Delta \mid U_\Delta \leq N_{\mathbb{E}}(A(X))\rangle$, where Δ is a root base.

Proposition 3.2. *[1] The group* $H(A(X))$ *is the semidirect product* $R \rtimes L$, *where*

$L = \langle r_\Delta(k) \mid s_\Delta = x + y$ for $x, y \in X\rangle$ called the Levi-complement and $R = \langle r_\Delta(k) \mid \Delta \cap X = \emptyset\rangle$ called the unipotent radical.

Theorem 3.2. *Let Δ be a root base and $U = A(\Delta) = \langle e_x | x \in \Delta\rangle$. If $H = H(A(\Delta)) = R \rtimes L \leq N_{\mathbb{E}}(A(\Delta))$, then*

1. *$(A(\Delta^*))^H = A(\Delta^*)^R$, where $\Delta^* = \Delta^{\sigma_s}$, $s = s_\Delta$.*
2. *R acts regularly on $A(\Delta^*)$; that is, $N_R(A(\Delta^*)) = I$.*
3. *If $|K| = q$, then $H \cong q^{1+20} \rtimes SL_6(q)$ the maximal parabolic subgroup P_6 in $E_6(K)$.*

Proof. R acts trivially on $A(\Delta)$ and leaves the sequence $0 < A(\Delta) < A(\Delta \cup \Delta_0) < A$ invariant. Hence $A(\Delta^*)^H = A(\Delta^*)^R$ as L stabilizes $A(\Delta^*)$. So it remains to show that $N_R(A(\Delta^*)) = I$. Let $g \in N_R(A(\Delta^*))$. This implies that $e_x^g = e_x$, $\forall x \in \Delta$ and also
$e_x^g = e_x$, $\forall x \in \Delta^*$, because if $x \in \Delta^*$, then $e_x^g - e_x \in A(\Delta \cup \Delta_0) \cap A(\Delta^*) = 0$ or $e_x^g = e_x$ as $[A, R] \leq A(\Delta \cup \Delta_0)$. If $z \in \Delta_0$, then $z = x + y = s_\Delta$, $x, y \in \Delta$, $x \neq y$. If $y_1 \in \Delta \setminus \{x, y\}$, then $(y_1|z) = (y_1 + s|s) = 0$ and $e_z^g = e_z + u$ for some $u \in A(\Delta)$. Hence $e_{y_1+s_\Delta}(e_z + u) = e^g_{y_1+s_\Delta} e_z^g = (e_{y_1+s_\Delta} e_z)^{g^*} = 0$. This implies $0 = e_{y_1+s_\Delta}(e_z + u) = e_{y_1+s_\Delta} u$. Hence $e_{y_1+s_\Delta} u = 0$, $\forall y_1 \in \Delta \setminus \{x, y\}$, which implies $u = 0$. Hence $g = I$. This proves (1) and (2).

Let $R_0 = \{\Delta^*\}$ and R_1 be the set of all root bases Γ such that $|\Gamma \cap \Delta^*| = 3$. By [2, Corollary 6.2], $|\Delta^* \cap \Gamma| = 3$ if $|\Delta \cap \Gamma| = \emptyset$. This implies that $|R_1| = \binom{6}{3} = 20$ and $R = \{r_\Gamma(k) \mid \Gamma \in R_0 \cup R_1 \ , \ k \in K\}$. Hence, the unipotent radical $R \cong K^{1+20}$. The Levi-complement L leaves $A(\Delta)$, $A(\Delta^*)$ and $A(\Delta_0)$ invariant and acts faithfully on $A(\Delta)$, induces $SL(A(\Delta)) \cong SL_6(K)$ on $A(\Delta)$. As R acts regularly on $A(\Delta^*)$ and if $|K| = q$ a power of prime, then $|(A(\Delta^*))^H| = |(A(\Delta^*))^R| = |R| = q^{1+20}$. Hence $H(A(\Delta)) = q^{1+20} \rtimes SL_6(q)$, which is the maximal parabolic subgroup P_6 in $E_6(q)$. This proves (3). □

Remark 3.2. *The space $A(\Delta)$ of dimension 6 is called a Tits subspace in V_6 and the group $H(A(\Delta))$ computed above is a Borel subgroup of $E_6(K)$, that is the stabilizer of $A(\Delta)$ in $E_6(K)$.*

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