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Characterizations of $\delta s(\Lambda, s)$-symmetric spaces and sober $\delta s(\Lambda, s)$-R_0 spaces

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Abstract

This paper is concerned with the concepts of $\delta s(\Lambda, s)$-symmetric spaces and sober $\delta s(\Lambda, s)$-R_0 spaces. Moreover, some characterizations of $\delta s(\Lambda, s)$-symmetric spaces and sober $\delta s(\Lambda, s)$-R_0 spaces are established.

1 Introduction

Semi-open sets, preopen sets and δ-open sets play an important role in the theory of classical point set topology. In 1963, Levine [6] offered a new concept in the field of topology by introducing the notion of semi-open sets in topological spaces. In 1968, Veličko [10] introduced δ-open sets, which are stronger than open sets. In 1997, Park et al. [7] introduced δ-semiopen sets which are stronger than semi-open sets but weaker than δ-open sets. In 2003, Caldas et al. [3] introduced some weak separation axioms by utilizing δ-semiopen sets and the δ-semiclosure operator. Moreover, Caldas et al. [3] investigated some characterizations of sober δ-semiR_0 spaces. Caldas and

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Jafari [4] introduced and studied the notion of Λ_δ-symmetric spaces. In 2005, Caldas et al. [2] investigated the notion of δ-Λ_s-semiclosed sets which is defined as the intersection of a δ-Λ_s-set and a δ-semiclosed set. In [1], the present authors introduced and investigated the concept of (Λ, s)-closed sets by utilizing the notions of Λ_s-sets and semi-closed sets. Quit recently, Pue-on and Boonpok [8] introduced and studied the notions of $\delta s(\Lambda, s)$-open sets and $\delta s(\Lambda, s)$-closed sets. In this paper, we introduce the concepts of $\delta s(\Lambda, s)$-symmetric spaces and sober $\delta s(\Lambda, s)$-R_0 spaces. Moreover, some characterizations of $\delta s(\Lambda, s)$-symmetric spaces and sober $\delta s(\Lambda, s)$-R_0 spaces are investigated.

2 Preliminaries

Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X, τ) is called *semi-open* [6] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$. A subset A of a topological space (X, τ) is called (Λ, s)-*closed* [1] if $A = T \cap C$, where T is a Λ_s-set and C is a semi-closed set. The complement of a (Λ, s)-closed set is called (Λ, s)-*open*. Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a (Λ, s)-*cluster point* [1] of A if $A \cap U \neq \emptyset$ for every (Λ, s)-open set U of X containing x. The set of all (Λ, s)-cluster points of A is called the (Λ, s)-*closure* [1] of A and is denoted by $A^{(\Lambda,s)}$. The union of all (Λ, s)-open sets contained in A is called the (Λ, s)-*interior* [1] of A and is denoted by $A_{(\Lambda,s)}$. A point x of X is called a $\delta(\Lambda, s)$-*cluster point* [9] of A if $A \cap [V^{(\Lambda,s)}]_{(\Lambda,s)} \neq \emptyset$ for every (Λ, s)-open set V of X containing x. The set of all $\delta(\Lambda, s)$-cluster points of A is called the $\delta(\Lambda, s)$-*closure* [9] of A and is denoted by $A^{\delta(\Lambda,s)}$. If $A = A^{\delta(\Lambda,s)}$, then A is said to be $\delta(\Lambda, s)$-*closed* [9]. The complement of a $\delta(\Lambda, s)$-closed set is said to be $\delta(\Lambda, s)$-*open* [9]. The union of all $\delta(\Lambda, s)$-open sets contained in A is called the $\delta(\Lambda, s)$-*interior* [9] of A and is denoted by $A_{\delta(\Lambda,s)}$. A subset A of a topological space (X, τ) is said to be $\delta s(\Lambda, s)$-*open* [8] if $A \subseteq [A_{(\Lambda,s)}]^{\delta(\Lambda,s)}$. The complement of a $\delta s(\Lambda, s)$-open set is said to be $\delta s(\Lambda, s)$-*closed*. The family of all $\delta s(\Lambda, s)$-open (resp. $\delta s(\Lambda, s)$-closed) sets in a topological space (X, τ) is denoted by $\delta s(\Lambda, s)O(X, \tau)$ (resp. $\delta s(\Lambda, s)C(X, \tau)$). A subset N of a topological space (X, τ) is called a $\delta s(\Lambda, s)$-*neighborhood* [8] of a point $x \in X$ if there exists a $\delta s(\Lambda, s)$-open set V such that $x \in V \subseteq N$. Let A be a subset of a topological space (X, τ). A point x of X is called a $\delta s(\Lambda, s)$-*cluster point* [8] of A if $A \cap U \neq \emptyset$ for every $\delta s(\Lambda, s)$-open set U of X containing x. The set of all $\delta s(\Lambda, s)$-cluster points of A is called the $\delta s(\Lambda, s)$-*closure* [8] of A and is denoted by $A^{\delta s(\Lambda,s)}$. A subset $\delta s(\Lambda, s)Ker(A)$ is defined as follows: $\delta s(\Lambda, s)Ker(A) = \cap\{U \mid A \subseteq U, U \in \delta s(\Lambda, s)O(X, \tau)\}$ [8].

3 On characterizations of $\delta s(\Lambda, s)$-symmetric spaces

In this section, we introduce the notion of $\delta s(\Lambda, s)$-symmetric spaces. Moreover, we discuss several characterizations of $\delta s(\Lambda, s)$-symmetric spaces.

Definition 3.1. *A topological space (X, τ) is said to be $\delta s(\Lambda, s)$-symmetric if for each x and y in X, $x \in \{y\}^{\delta s(\Lambda,s)}$ implies $y \in \{y\}^{\delta s(\Lambda,s)}$.*

Definition 3.2. *A subset A of a topological space (X, τ) is said to be generalized $\delta s(\Lambda, s)$-closed (briefly, g-$\delta s(\Lambda, s)$-closed) if $A^{\delta s(\Lambda,s)} \subseteq U$ whenever $A \subseteq U$ and U is $\delta s(\Lambda, s)$-open in (X, τ).*

Theorem 3.3. *A subset A of a topological space (X, τ) is g-$\delta s(\Lambda, s)$-closed if and only if $F \cap A^{\delta s(\Lambda,s)} = \emptyset$ whenever $A \cap F = \emptyset$ and F is $\delta s(\Lambda, s)$-closed.*

Proof. The proof follows from Theorem 16 of [1]. □

Theorem 3.4. *A subset A of a topological space (X, τ) is g-$\delta s(\Lambda, s)$-closed if and only if $A \cap \{x\}^{\delta s(\Lambda,s)} \neq \emptyset$ for every $x \in A^{\delta s(\Lambda,s)}$.*

Proof. The proof follows from Theorem 17 of [1]. □

Theorem 3.5. *A topological space (X, τ) is $\delta s(\Lambda, s)$-symmetric if and only if $\{x\}$ is g-$\delta s(\Lambda, s)$-closed for each $x \in X$.*

Proof. Assume that $x \in \{y\}^{\delta s(\Lambda,s)}$ but $y \in \{x\}^{\delta s \Lambda,s)}$. This means that the complement of $\{x\}^{\delta s(\Lambda,s)}$ contains y. Thus, the set $\{y\}$ is a subset of the complement of $\{x\}^{\delta s(\Lambda,s)}$. This implies that $\{y\}^{\delta s(\Lambda,s)}$ is a subset of the complement of $\{x\}^{\delta s(\Lambda,s)}$. Now the complement of $\{x\}^{\delta s(\Lambda,s)}$ contains x which is a contradiction.

Conversely, suppose that $\{x\} \subseteq U \in \delta s(\Lambda, s)O(X, \tau)$, but $\{x\}^{\delta s(\Lambda,s)}$ is not a subset of U. This means that $\{x\}^{\delta s(\Lambda,s)}$ and the complement of U are not disjoint. Let y belongs to their intersection. Now we have $x \in \{y\}^{\delta s(\Lambda,s)}$ which is a subset of the complement of U and $x \notin U$. This is a contradiction. □

Definition 3.6. *A topological space* (X, τ) *is called* $\delta s(\Lambda, s)$-T_1 *if for any distinct pair of points* x *and* y *in* X*, there exist a* $\delta s(\Lambda, s)$*-open set* U *of* X *containing* x *but not* y *and a* $\delta s(\Lambda, s)$*-open set* V *of* X *containing* y *but not* x.

Theorem 3.7. *A topological space* (X, τ) *is* $\delta s(\Lambda, s)$-T_1 *if and only if the singleton are* $\delta s(\Lambda, s)$*-closed sets.*

Proof. The proof follows from Theorem 3.2 of [5]. □

Theorem 3.8. *If a topological space* (X, τ) *is* $\delta s(\Lambda, s)$-T_1*, then* (X, τ) *is* $\delta s(\Lambda, s)$*-symmetric.*

Proof. Let $x \in X$. Since (X, τ) is $\delta s(\Lambda, s)$-T_1, $\{x\}$ is $\delta s(\Lambda, s)$-closed. Then, $\{x\}$ is g-$\delta s(\Lambda, s)$-closed and by Theorem 3.5, (X, τ) is $\delta s(\Lambda, s)$-symmetric. □

Definition 3.9. *A topological space* (X, τ) *is called* $\delta s(\Lambda, s)$-T_0 *if for any distinct pair of points in* X*, there exists a* $\delta s(\Lambda, s)$*-open set containing one of the points but not the other.*

Remark 3.10. *If a topological space* (X, τ) *is* $\delta s(\Lambda, s)$-T_1*, then* (X, τ) *is* $\delta s(\Lambda, s)$-T_0.

Theorem 3.11. *For a topological space* (X, τ)*, the following properties are equivalent:*

(1) (X, τ) *is* $\delta s(\Lambda, s)$*-symmetric and* $\delta s(\Lambda, s)$-T_0*;*

(2) (X, τ) *is* $\delta s(\Lambda, s)$-T_1.

Proof. By Theorem 3.8 and Remark 3.10 it suffices to prove only (1) ⇒ (2). Let $x, y \in X$, $x \neq y$ and by $\delta s(\Lambda, s)$-T_0, we may assume that $x \in U \subseteq X - \{y\}$ for some $U \in \delta s(\Lambda, s)O(X, \tau)$. Then, $x \notin \{y\}^{\delta s(\Lambda,s)}$. Thus, $y \notin \{x\}^{\delta s(\Lambda,s)}$. There exists $V \in \delta s(\Lambda, s)O(X, \tau)$ such that $y \in V \subseteq X - \{x\}$. This shows that (X, τ) is $\delta s(\Lambda, s)$-T_1. □

Theorem 3.12. *For a* $\delta s(\Lambda, s)$*-symmetric space* (X, τ)*, the following properties are equivalent:*

(1) (X, τ) *is* $\delta s(\Lambda, s)$-T_0*;*

(2) (X, τ) *is* $\delta s(\Lambda, s)$-T_1.

Proof. (1) ⇒ (2): Follows from Theorem 3.11.
(2) ⇒ (1): Follows from Remark 3.10. □

4 Characterizations of sober $\delta s(\Lambda, s)$-R_0 spaces

In this section, we introduce the notion of sober $\delta s(\Lambda, s)$-R_0 spaces. Moreover, some characterizations of sober $\delta s(\Lambda, s)$-R_0 spaces are discussed.

Definition 4.1. *A topological space (X, τ) is said to be sober $\delta s(\Lambda, s)$-R_0 if $\bigcap\limits_{x \in X} \{x\}^{\delta s(\Lambda,s)} = \emptyset$.*

Theorem 4.2. *A topological space (X, τ) is sober $\delta s(\Lambda, s)$-R_0 if and only if $\delta s(\Lambda, s)Ker(\{x\}) \neq X$ for each $x \in X$.*

Proof. Suppose that the space (X, τ) is sober $\delta s(\Lambda, s)$-R_0. Assume that there exists a point y in X such that $\delta s(\Lambda, s)Ker(\{y\}) = X$. Let x be any point of X. Then, we have $x \in V$ for every $\delta s(\Lambda, s)$-open set V containing y and hence $y \in \{x\}^{\delta s(\Lambda,s)}$ for each $x \in X$. Thus, $y \in \bigcap\limits_{x \in X} \{x\}^{\delta s(\Lambda,s)}$. This is a contradiction.

Conversely, assume that $\delta s(\Lambda, s)Ker(\{x\}) \neq X$ for each $x \in X$. If there exists a point y in X such that $y \in \bigcap\limits_{x \in X} \{x\}^{\delta s(\Lambda,s)}$, then every $\delta s(\Lambda, s)$-open set containing y must contain every point of X. This implies that the space (X, τ) is the unique $\delta s(\Lambda, s)$-open set containing y. Thus, $\delta s(\Lambda, s)Ker(\{x\}) = X$ which is a contradiction. This shows that (X, τ) is sober $\delta s(\Lambda, s)$-R_0. ☐

Definition 4.3. *A function $f : (X, \tau) \to (Y, \sigma)$ is called $\delta s(\Lambda, s)$-closed if $f(F)$ is $\delta s(\Lambda, s)$-closed in Y for every $\delta s(\Lambda, s)$-closed set F of X.*

Theorem 4.4. *If $f : (X, \tau) \to (Y, \sigma)$ is an injective $\delta s(\Lambda, s)$-closed function and (X, τ) is sober $\delta s(\Lambda, s)$-R_0, then (Y, σ) is sober $\delta s(\Lambda, s)$-R_0.*

Proof. Since (X, τ) is sober $\delta s(\Lambda, s)$-R_0, $\bigcap\limits_{x \in X} \{x\}^{\delta s(\Lambda,s)} = \emptyset$. Since f is a $\delta s(\Lambda, s)$-closed injection, we have

$$\emptyset = f(\bigcap_{x \in X} \{x\}^{\delta s(\Lambda,s)}) = \bigcap_{x \in X} f(\{x\}^{\delta s(\Lambda,s)}) \supseteq \bigcap_{x \in X} \{f(x)\}^{\delta s(\Lambda,s)} \supseteq \bigcap_{y \in Y} \{y\}^{\delta s(\Lambda,s)}.$$

Thus, (Y, σ) is sober $\delta s(\Lambda, s)$-R_0. ☐

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References

[1] C. Boonpok, C. Viriyapong, On some forms of closed sets and related topics, Eur. J. Pure Appl. Math., **16,** no. 1, (2023), 336–362.

[2] M. Caldas, M. Ganster, D. N. Georgiou, S. Jafari, T. Noiri, δ-semiopen sets in topological spaces, Topology Proc., **29,** no. 2, (2005), 369–383.

[3] M. Caldas, D. N. Georgiou, S. Jafari, T. Noiri, More on δ-semiopen sets, Note di Matematica, **22,** no. 2, (2003), 1–14.

[4] M. Caldas, S. Jafari, On some low separation axioms in topological spaces, Houston J. Math., **29,** no. 1, (2003), 93–104.

[5] J. Khampakdee, C. Boonpok, On $\delta p(\Lambda, s)$-symmetric spaces, Int. J. Math. Comput. Sci., **18,** no. 4, (2023), 755–759.

[6] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, **70,** (1963), 36–41.

[7] J. H. Park, B. Y. Lee, M. J. Son, On δ-semiopen sets in topological spaces, J. Indian Acad. Math., **19,** (1997), 59–67.

[8] P. Pue-on, C. Boonpok, On $\delta s(\Lambda, s)$-open sets in topological spaces, Int. J. Math. Comput. Sci., **18,** no. 4, (2023), 749–753.

[9] N. Srisarakham, C. Boonpok, On characterizations of $\delta p(\Lambda, s)$-$\mathscr{D}_1$ spaces, Int. J. Math. Comput. Sci., **18,** no. 4, (2023), 743–747.

[10] N. V. Veličko, H-closed topological spaces, Amer. Math. Soc. Transl., **78,** no. 2, (1968), 102–118.