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# On the Exponential Diophantine equation $11^x - 17^y = z^2$

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**Abstract**

Let $x, y$ and $z$ be non-negative integers. Using the factoring method, the order of a modulo, and the division algorithm, we show that the exponential Diophantine equation $11^x - 17^y = z^2$ has the unique solution $(x, y, z) = (0, 0, 0)$.

## 1 Introduction

Exponential Diophantine equations have at least one unknown exponent variable. For over a decade, mathematical researchers have been interested in discussing the existence of integer solutions of exponential Diophantine equations. For example, Acu [1] proved that the exponential Diophantine equation $2^x + 5^y = z^2$ has only two non-negative integer solutions, $(x, y, z) \in \{(3, 0, 3), (2, 1, 3)\}$. After that, Suvarnamani et al. [9] studied the two Diophantine equations, including $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ found that neither equation has a non-negatie integer solution. During the period 2012-2017, there were many research articles on the exponential Diophantine

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equation [[2], [4], [6], and [8]]. In 2018, Rabago [7] studied $4^x - p^y = 3z^2$ where $p$ is prime, and he determined all solutions of the equations $4^x - 7^y = z^2$ and $4^x - 11^y = z^2$. After that, Brushtein [3] suggested that $6^x - 11^y = z^2$ has no solution when $2 \leqslant x \leqslant 16$. More recently, Thongnak et al. [[10], [11], [12]] solved three equations including $7^x - 5^y = z^2$, $7^x - 2^y = z^2$ and $15^x - 13^y = z^2$. These equations motivated us to study other equations.

In this study, we show that the trivial solution is the only non-negative integer solution of the exponential Diophantine equation $11^x - 17^y = z^2$.

## 2 Preliminaries

In this section, we give an important definition and a theorem used in our proof.

**Definition 2.1.** *[5] If $n$ is a positive integer and $\gcd(a, n) = 1$, the least positive integer $k$ such that $a^k \equiv 1 \pmod{n}$ is called the order of $a$ modulo $n$ and is denoted by $\mathrm{ord}_n a$.*

**Theorem 2.2.** *(Euler's criterion) [5] Let $p$ be an odd prime and $\gcd(a, p) = 1$. Then $a$ is a quadratic residue of $p$ if and only if $a^{(p-1)/2} \equiv 1 \pmod{p}$.*

**Theorem 2.3.** *[5] Let the integer $a$ have order $k$ modulo $n$. Then $a^h \equiv 1 \pmod{n}$ if and only if $k|h$; in particular, $k|\phi(n)$.*

**Theorem 2.4.** *(Euclid' s Lemma) [5] If $a|bc$ and $\gcd(a, b) = 1$, then $a|c$.*

**Theorem 2.5.** *[5] If $a|c$, $b|c$ and $\gcd(a, b) = 1$, then $ab|c$.*

## 3 Main result

**Theorem 3.1.** *The exponential Diophantine equation $11^x - 17^y = z^2$ has the unique solution $(x, y, z) = (0, 0, 0)$ where $x, y$ and $z$ are non-negative integers.*

*Proof.* Let $x, y$ and $z$ be non-negative integers such that

$$11^x - 17^y = z^2. \tag{3.1}$$

We will consider four cases as follows.
Case 1, $x = y = 0$. From (3.1), we obtain $z^2 = 0$ or $z = 0$. Therefore, one solution to the equation is $(x, y, z) = (0, 0, 0)$.
Case 2, $x = 0$ and $y > 0$, (3.1) becomes $z^2 = 1 - 17^y < 0$, which is impossible.

Case 3, $x > 0$ and $y = 0$. (3.1) becomes $z^2 = 11^x - 1$, implying that $z^2 \equiv 10 \pmod{11}$ which is impossible because $z^2 \equiv 0, 1, 3, 4, 5, 9 \pmod{11}$.
Case 4, $x > 0$ and $y > 0$. From (3.1), we have $z^2 \equiv 11^x \pmod{17}$. So $11^x$ is a quadratic residue of 17. By Theorem 2.2, it follows that $11^{8x} \equiv 1 \pmod{17}$ and so $(-1)^x \equiv 1 \pmod{17}$. Clearly, $x$ must be even. By substituting $x = 2k$, $\exists k \in \mathbb{Z}^+$ into (3.1), we obtain $17^y = 11^{2k} - z^2$. Factoring, we get $17^y = (11^k - z)(11^k + z)$. There exists $\alpha \in \{0, 1, 2, ..., y\}$ such that $11^k - z = 17^\alpha$ and $11^k + z = 17^{y-\alpha}$, where $\alpha < y - \alpha$. Both equations imply that $2 \cdot 11^k = 17^\alpha + 17^{y-\alpha}$ or $2 \cdot 11^k = 17^\alpha(1 + 17^{y-2\alpha})$. Since $17 \nmid 2 \cdot 11^k$, we have $\alpha = 0$. As a result,

$$2 \cdot 11^k = 1 + 17^y. \tag{3.2}$$

(3.2) implies that $2 \equiv 1 + 2^y \pmod{5}$ or $2^y \equiv 1 \pmod{5}$. Since $\mathrm{ord}_5 2 = 4$, we have $4|y$. Let $y = 4l$, $\exists\, l \in \mathbb{Z}^+$. By (3.2), we have $6^{4l} \equiv -1 \pmod{11}$, which implies that $6^{4l} \equiv 6^5 \pmod{11}$. Clearly, $\mathrm{ord}_{11} 6 = 10$, so $4l \equiv 5 \pmod{10}$. There exists $t \in \mathbb{Z}^+$ such that $4l = 5 + 10t$, which yields $2(2l - 5t) = 5$ which impossible. Combining all cases, we have shown that $(x, y, z) = (0, 0, 0)$ is the unique solution to the equation. □

# 4 Conclusion

In this article, we have shown that the exponential Diophantine equation $11^x - 17^y = z^2$ where $x, y$ and $z$ are non-negative integers has the unique solution $(x, y, z) = (0, 0, 0)$.

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