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Some characterizations of regular duo Γ-semigroups

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Abstract

It is well known that a Γ-semigroup as a generalization of a semigroup. In this article, we present some characterizations of the regularity of duo Γ-semigroups by the properties of left Γ-ideals, right Γ-ideals, quasi-Γ-ideals and bi-Γ-ideals of Γ-semigroups.

1 Introduction

In 1971, Lajos [9] examined the characterizations of regular duo semigroups. The notion of Γ-semigroups, a generalization of semigroups, was introduced by Sen [13]. The study of regularity of Γ-semigroups has been investigated continuously, for example, in regular Γ-semigroups [1, 14], in completely regular Γ-semigroups [10, 15], in weakly regular Γ-semigroups [8]. In 2008, Hila [5] characterized regular duo Γ-semigroups. Moreover, the characterizations of regularities in duo Γ-semigroups were considered by Rao at el. [11] in 2012. Furthermore, Kehayopulu [6] studied the class of regular duo po-Γ-semigroups, and then, in 2021, Luangchaisri and Changphas [7] characterized completely regular 2-duo semigroups by means of $(2, 0)$-ideals, $(0, 2)$-ideals,

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$(2,2)$-ideals and $(2,2)$-quasi-ideals of semigroups. In this paper, we give some characterizations of regular duo Γ-semigroups using left Γ-ideals, right Γ-ideals, quasi-Γ-ideals and bi-Γ-ideals of Γ-semigroups.

2 Preliminaries

Let S and Γ be nonempty sets. Then S is called a Γ-*semigroup* [13] if there exists a mapping $S \times \Gamma \times S \to S$ (the image of (x, α, y) written as $x\alpha y$, for all $x, y \in S$ and $\alpha \in \Gamma$) satisfying $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. For any nonempty subsets A and B of a Γ-semigroup S, the product of A and B is defined as follows:

$$A\Gamma B = \{a\alpha b \mid a \in A, b \in B, \alpha \in \Gamma\}.$$

For every subset $\{a\}$ of S, we write $A\Gamma a$ for $A\Gamma\{a\}$ and $a\Gamma A$ for $\{a\}\Gamma A$. In addition, the set $A\Gamma A$ denoted by A^2.

The following concepts are contained in [2, 3, 4]. Let S be a Γ-semigroup and A be a nonempty subsets of S. Then:
(i) A is called a Γ-*subsemigroup* of S if $A^2 \subseteq A$;
(ii) A is called a *left* (resp., *right*) Γ-*ideal* of S if $S\Gamma A \subseteq A$ (resp., $A\Gamma S \subseteq A$);
(iii) A is called a *two-sided* Γ-*ideal* of S if it is both a left and a right Γ-ideal of S;
(iv) A is known to be a *quasi*-Γ-*ideal* of S if $S\Gamma A \cap A\Gamma S \subseteq A$;
(v) A is said to be a *bi*-Γ-*ideal* of S if $A^2 \subseteq A$ and $A\Gamma S\Gamma A \subseteq A$. For a Γ-semigroup S, we obtain that every left (resp., right) Γ-ideal of S is also a quasi-Γ-ideal of S, while every quasi-Γ-ideal of S is also a bi-Γ-ideal of S.

Let S be a Γ-semigroup and A be a nonempty subset of S. The *left* (resp., *right*) Γ-*ideal of* S *generated by* A [12] is defined as the form

$$(A)_l = A \cup S\Gamma A \text{ (resp., } (A)_r = A \cup A\Gamma S).$$

If $A = \{a\}$, then we denote $(\{a\})_l = (a)_l$ and $(\{a\})_r = (a)_r$. A Γ-semigroup S is called *duo* (see, [5]) if for every one-sided Γ-ideal of S is a two-sided Γ-ideal of S. We observe that if A is any nonempty subset of a duo Γ-semigroup S, then $(A)_l = (A)_r$.

3 Regular duo Γ-semigroups

In this section, we will present the results concerning the characterizations of regular duo Γ-semigroups using the concepts of different types of their Γ-ideals.

Let S be a Γ-semigroup. Then the element a of S is called *regular* (cf. [5, 15]) if there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. The Γ-semigroup S is said to be *regular* if for every element of S is regular. Equivalent Definitions:
(i) $a \in a\Gamma S\Gamma a$ for all $a \in S$;
(ii) $A \subseteq A\Gamma S\Gamma A$ for all nonempty subset A of S.

Theorem 3.1. *Let S be a Γ-semigroup. Then S is a regular duo Γ-semigroup if and only if the following conditions hold:*

(i) $(L \cup L\Gamma S)^2 = L$, *for every left Γ-ideal L of S;*

(ii) $(R \cup S\Gamma R)^2 = R$, *for every right Γ-ideal R of S.*

Proof. Assume that S is a regular duo Γ-semigroup. Let L be a left Γ-ideal of S. Then $L \subseteq L\Gamma(S\Gamma L) \subseteq L\Gamma L \subseteq S\Gamma L \subseteq L$. So, $L = L\Gamma L$. By assumption, L is also a right Γ-ideal of S. It follows that

$$\begin{aligned} L = L\Gamma L = (L\Gamma L)\Gamma(L\Gamma L) = (L\Gamma L)^2 = (L\Gamma L \cup L\Gamma L)^2 \\ \subseteq (L \cup L\Gamma S)^2 \subseteq (L \cup L)^2 = L\Gamma L = L. \end{aligned}$$

This means that $(L \cup L\Gamma S)^2 = L$. In the other case, we can proved similarly.

Conversely, let L be a left Γ-ideal of S. By assumption, we have

$$\begin{aligned} L\Gamma S &= (L \cup L\Gamma S)^2\Gamma S \subseteq (L \cup L\Gamma S)\Gamma(L\Gamma S \cup L\Gamma S) \\ &= (L \cup L\Gamma S)\Gamma(L\Gamma S) \subseteq (L \cup L\Gamma S)\Gamma(L \cup L\Gamma S) \\ &= (L \cup L\Gamma S)^2 = L. \end{aligned}$$

Hence, L is a right Γ-ideal of S. In the same way, if R is a right Γ-ideal of S, then using (ii) we have that R is also a left Γ-ideal of S. Thus, S is duo. Next, we want to show that S is regular. Let $a \in S$. Then,

$$\begin{aligned} a \in (a)_r = ((a)_r \cup (a)_r\Gamma S)^2 &\subseteq ((a)_r \cup (a)_r)^2 \\ &= (a)_r\Gamma(a)_r = (a)_r\Gamma(a)_l \\ &= (a \cup a\Gamma S)\Gamma(a \cup S\Gamma a) \\ &\subseteq a\Gamma a \cup a\Gamma S\Gamma a. \end{aligned}$$

Consequently, S is regular. □

Theorem 3.2. *Let S be a Γ-semigroup. Then S is a regular duo Γ-semigroup if and only if $(B \cup S\Gamma B)^2 = B = (B \cup B\Gamma S)^2$, for each bi-$\Gamma$-ideal B of S.*

Proof. Let B be any bi-Γ-ideal B of S. Now, consider

$$\begin{aligned}(B \cup B\Gamma S)^2 &\subseteq B\Gamma B \cup B\Gamma B\Gamma S \cup B\Gamma S\Gamma B \cup B\Gamma S\Gamma B\Gamma S\\ &\subseteq B\Gamma S \subseteq (B\Gamma S\Gamma B)\Gamma S\\ &\subseteq B\Gamma S\Gamma(B\Gamma S\Gamma B)\Gamma S.\end{aligned}$$

It is easy to see that $S\Gamma B$ is a left Γ-ideal of S. By the hypothesis, we obtain that $S\Gamma B$ is also a right Γ-ideal of S. Thus

$$B\Gamma S\Gamma B\Gamma S\Gamma B\Gamma S = (B\Gamma S\Gamma B)\Gamma((S\Gamma B)\Gamma S) \subseteq B\Gamma S\Gamma B \subseteq B.$$

Hence $(B \cup B\Gamma S)^2 \subseteq B$. On the other hand,

$$B \subseteq (B)_r \subseteq (B)_r\Gamma S\Gamma(B)_r \subseteq (B)_r\Gamma(B)_r = (B \cup B\Gamma S)^2.$$

Therefore, $B = (B \cup B\Gamma S)^2$. Similarly, we can show that $B = (B \cup S\Gamma B)^2$. This shows that $(B \cup S\Gamma B)^2 = B = (B \cup B\Gamma S)^2$.

Conversely, let L be a left Γ-ideal of S. Also, L is a bi-Γ-ideal of S. By the assumption, $L = (L \cup L\Gamma S)^2$. At the same time, for any right Γ-ideal R of S, $R = (R \cup S\Gamma R)^2$. By Theorem 3.1, we conclude that S is a regular duo Γ-semigroup. □

Theorem 3.3. *Let S be a Γ-semigroup. Then S is a regular duo Γ-semigroup if and only if $(Q \cup Q\Gamma S)^2 = Q = (Q \cup S\Gamma Q)^2$, for any quasi-$\Gamma$-ideal Q of S.*

Proof. Assume that S is a regular duo Γ-semigroup. Let Q be a quasi-Γ-ideal of S. Then $(Q \cup Q\Gamma S)^2 \subseteq Q\Gamma Q \cup Q\Gamma Q\Gamma S \cup Q\Gamma S\Gamma Q \cup Q\Gamma S\Gamma Q\Gamma S \subseteq Q\Gamma S$. Since S is duo and $S\Gamma Q$ is a left Γ-ideal of S, it follows that

$$\begin{aligned}(Q \cup Q\Gamma S)^2 &\subseteq Q\Gamma Q \cup Q\Gamma Q\Gamma S \cup Q\Gamma S\Gamma Q \cup Q\Gamma S\Gamma Q\Gamma S\\ &\subseteq S\Gamma Q \cup (S\Gamma Q)\Gamma S\\ &\subseteq S\Gamma Q \cup S\Gamma Q = S\Gamma Q.\end{aligned}$$

This implies that $(Q \cup Q\Gamma S)^2 \subseteq Q\Gamma S \cap S\Gamma Q \subseteq Q$. As shown in the proof of Theorem 3.2, $Q \subseteq (Q \cup Q\Gamma S)^2$. Hence, $(Q \cup Q\Gamma S)^2 = Q$. Similarly, we can prove that $(Q \cup S\Gamma Q)^2 = Q$. Therefore, $(Q \cup Q\Gamma S)^2 = Q = (Q \cup S\Gamma Q)^2$.

Conversely, let L and R be any left Γ-ideal and right Γ-ideal of S, respectively. Then, L and R are quasi-Γ-ideals of S. By using the hypothesis, we have that $L = (L \cup L\Gamma S)^2$ and $R = (R \cup S\Gamma R)^2$. By Theorem 3.1, it turns out that S is a regular duo Γ-semigroup. □

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