International Journal of Mathematics and
Computer Science, **19**(2024), no. 2, 311–313

Hopfcity and noncosingular submodules

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(Received July 25, 2023, Accepted September 15, 2023,
Published November 10, 2023)

Abstract

The analysis of modules via their ring of homomorphisms has been of great interest in module theory.
In this paper, we present the concept of γ-weakly Hopfian. Then we look at properties of γ-weakly Hopfian including the following: Let X be a module satisfying the ascending chain conditions on γ-small submodules. Then X is γ-wH.

1 Introduction

A module X is called Hopfian if each its epimorphism is an automorphism and X is called co-Hopfian if any of its injective endomorphism is an automorphism. A submodule G of X is said to be γ-small in X ($G \ll_\gamma X$), if whenever $X = G + L$ with X/L noncosingular, then we obtain $L = X$ [4]. In [3], the notion of γ-Hopfian was investigated. A module X is called γ-Hopfian if the kernel of each its epimorphism is γ-small. In [1], the notion of δ-weakly Hopfian was introduced. We say that X is δ-weakly Hopfian if each of its δ-small epimorphism is a monomorphism. Such modules have been considered by many authors [1, 2, 3, 4].

Key words and phrases: Hopfian module, γ-weakly Hopfian module, γ-Hopfian module.
AMS (MOS) Subject Classifications: 16D10, 16D40, 16D90.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

Based on the cited work,we present the concept of γ-weakly Hopfian (γ-wH modules, for short). We call X γ-wH module if every γ-small epimorphism of X is a monomorphism.

2 Modules whose γ-small epimorphisms are isomorphism

Definition 2.1. *We say that an R-module X is γ-weakly Hopfian (γ-wH modules, for short) if each γ-small epimorphism of X is a monomorphism.*

The following example shows an R-module which is not γ-wH.

Example 2.2. *Consider the Z-module Z_{p^∞}. As in Z_{p^∞}, any proper subgroup is γ-small. Then the kernel of any epimorphism of Z_{p^∞} is γ-small, but the multiplication by p is a surjective endomorphism of Z_{p^∞} not a monomorphism.*

Proposition 2.3. *Each direct summand of a γ-wH R-module X is γ-wH.*

Proof.
Consider $P \leq^{\oplus} X$. There is a submodule Q of X with $X = P \oplus Q$. Let $h : P \to P$ be a γ-small epimorphism. Then h induces an epimorphism $h \oplus 1_Q : X \to X$ with $(h \oplus 1_Q)(p + q) = h(p) + q$, with $p \in P$ and $q \in Q$. From [4], $Ker(h \oplus 1_Q) = Ker(h) \oplus 0 \ll_\gamma P \oplus Q$. As X is γ-wH, $h \oplus 1_Q$ is a bijective endomorphism of X and so h is a a bijective endomorphism of P.

Theorem 2.4. *Let $X = X_1 \oplus X_2$ be a module such that X_1 and X_2 are fully invariant submodules over each surjection of X. Then X is γ-wH iff X_1, X_2 are γ-wH.*

Proof.
$\Rightarrow$) This follows from Proposition 2.3.

$\Leftarrow$) Let $f : X \to X$ be a γ-small epimorphism. Then $f|_{X_i} : X_i \to X_i$ where $i \in \{1; 2\}$ is a γ-small surjective endomorphism. By the hypothesis, $f|_{X_i}$ is a bijective endomorphism. Let $f(y_1 + y_2) = 0$. Then $f(y_1) + f(y_2) = 0$ and so $y_1 = y_2 = 0$. Thus f is a monomorphism. Consequently, X is γ-wH.

We say that X is a duo module if any its submodule is fully invariant.

Corollary 2.5. *Let $X = X_1 \oplus X_2$ be a duo R-module. Then X is γ-wH iff X_1 and X_2 are γ-wH.*

Lemma 2.6. *Let X, B, and C be modules. If two surjective homomorphisms $k : X \to B$ and $h : B \to C$ are γ-small, then hk is γ-small.*

Proof.
Let X, B, and C be modules and let $k : X \to B$ and $h : B \to C$ be two γ-small epimorphisms. Suppose that $Kerhk + P = X$, where $P \leq X$ with X/P is noncosingular. Then $hk(P) = hk(X.)$ Hence $k(P) + Kerh = k(X)$. Since X/P is noncosingular, $k(X)/k(P)$ is noncosingular. Thus $k(P) = k(X)$ implies $P + Kerk = X$. As a result, $P = X$ and hk is γ-small.

Theorem 2.7. *Let X be an R-module satisfying the ascending chain conditions on γ-small submodules. Then X is γ-wH.*

Proof.
Let $f : X \to X$ be a γ-small surjective endomorphism of an R-module X. By Lemma 2.6, $Kerf \subseteq Kerf^2 \subseteq ... \subseteq Kerf^n \subseteq ...$ is an ascending chain of γ-small. As X satisfies the ascending chain conditions of γ-small submodules, there exists $n \geq 0$ with $Kerf^n = Kerf^{n+1}$. Let $x \in Kerf$. Then $f(x) = 0$. As f is surjective endomorphism, there is $x_1 \in X$ with $f(x_1) = x$. Since f is a surjective endomorphism, there is $x_2 \in X$ with $f(x_2) = x_1$. Repeating the process, we find that $x_{n-1} \in M$ with $f(x_n) = x_{n-1}$. Then $x = f(x_1) = f^2(x_2) = ... = f^n(x_n)$. Since $x \in Kerf$, we have $0 = f(x) = f(f^n(x_n))$; that is, $f^{n+1}(x_n) = 0$. Thus $x_n \in Kerf^{n+1} = Kerf^n$. So $f^n(x_n) = 0$. Then $x = 0$ and hence f is automorphism.

References

[1] A. El Moussaouy, M. Khoramdel, A. R. Moniri Hamzekolaee, M. Ziane, Weak Hopfcity and singular modules, Annali dell' Università di Ferrara, **68**, (2022), 69–78.

[2] A. El Moussaouy, A. R. Moniri Hamzekolaee, M. Ziane, Jacobson Hopfian Modules, Algebra and Discrete Mathematics, **33,** no. 1, (2022), 116–127.

[3] A. El Moussaouy, M. Ziane, Modules whose surjective endomorphisms have a γ-small kernel, Algebraic Structures and Their Applications, **9,** no. 2, (2022), 121–133.

[4] A. R. Moniri Hamzekolaee, An approach to H-supplemented modules via noncosingular modules, Ann. Univ. Ferrara, **67**, (2021), 111–120.