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Extension of φ-centralizers on semiprime rings

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Abstract

The objective of this research paper is to justify that an additive mapping $\digamma$ from a semiprime ring R to itself will be φ-centralizer having a suitable torsion restriction on R if it satisfies certain algebraic equations.

1 Introduction

Throughout this paper, R stands for an associative ring with unity e. A ring R is called p-torsion free, where p is positive integer, if $pr = 0$ implies $r = 0$ for all $r \in R$. A ring R is called prime if $rRt = \{0\}$ entails either $r = 0$ or $t = 0$, and is known as semiprime if $rRr = \{0\}$ yields $r = 0$. Helgosen [5] introduced the idea of centralizers on Banach algebras. A mapping $\digamma : R \to R$ is called a right (left) centralizer if it is additive and satisfies $\digamma(rt) = r\digamma(t)$

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$\big(F(rt) = F(r)t\big)$ for all $r, t \in R$. In particular, it is called a Jordan right (Jordan left) centralizer if $r = t$. If F is both a Jordan right centralizer and a Jordan left centralizer, then it is simply called a Jordan centralizer. Albas [1] called $F : R \to R$ is called a right (left) φ-centralizer if $F(rt) = \varphi(r)F(t)$ $\big(F(rt) = F(r)\varphi(t)\big)$ and it is additive, for every element $r, t \in R$ and F is known as a Jordan right (Jordan left) φ-centralizer if $F(r^2) = \varphi(r)F(r)$ $\big(F(r^2) = F(r)\varphi(r)\big)$ for all $r \in R$, where φ is an endomorphism defined on R. The mapping F is calleds a φ-centralizer, in case F is both right as well as left φ-centralizer. Every Jordan φ-centralizer is a φ-centralizer. However, the converse is not generally true. The converse of this statement is true under suitable torsion restriction on a semiprime ring [1]. For some recent extensions of the above results, the reader is referred to [2, 3]. Based on these findings, the authors of this work provide an extension of these mathematical assertions. More precisely, $F : R \to R$ will be a φ-centralizer, if F satisfies $3F(r^n s^n r^n) = F(r^n)\varphi(s^n r^n) + \varphi(r^n)F(s^n)\varphi(r^n) + \varphi(r^n s^n)F(r^n)$, for all r in a suitably torsion restricted semiprime ring R. The subsequent result is necessary to prove the fundamental theorem:

Lemma 1.1 ([4]). *Assume that R is a semiprime ring with the 2-torsion free condition and let $F : R \to R$ be an additive mapping satisfying the algebraic identity $2F(r^2) = F(r)\varphi(r) + \varphi(r)F(r)$ for all $r \in R$, where φ is a surjective endomorphism on R. Then F will be a φ-centralizer on R.*

2 Main result

Theorem 2.1. *Every additive mapping T from a $n!$-torsion free semiprime ring R to itself is a φ-centralizer if it satisfies the algebraic condition $3F(r^n s^n r^n) = F(r^n)\varphi(s^n r^n) + \varphi(r^n)F(s^n)\varphi(r^n) + \varphi(r^n s^n)F(r^n)$, $\forall r, s \in R$, where $n \geq 1$ is a fixed integer.*

Proof. Since

$$3F(r^n s^n r^n) = F(r^n)\varphi(s^n r^n) + \varphi(r^n)F(s^n)\varphi(r^n) + \varphi(r^n s^n)F(r^n), \quad (2.1)$$

$\forall r, s \in R$ then, in particular, choosing the identity e for r in (2.1), we find that

$$2F(s^n) = F(e)\varphi(s^n) + \varphi(s^n)F(e), \ \forall s \in R. \quad (2.2)$$

On the other hand, in particular, choosing the identity e for s in (2.1), we obtain

$$3F(r^{2n}) = F(r^n)\varphi(r^n) + \varphi(r^n)F(e)\varphi(r^n) + \varphi(r^n)F(r^n), \ \forall r \in R. \quad (2.3)$$

Next, replacing s by $ks+e$ in equation (2.2), we have

$$\sum_{i=0}^{n}\binom{n}{i}k^{n-i}[2F(s^{n-i})-\varphi(s^{n-i})F(e)-F(e)\varphi(s^{n-i})]=0\ \forall s\in R, k\in\mathbb{Z}^{+}.$$

Putting $k=1,2,3...,n-1$ one by one, we get a homogeneous system of $n-1$ linear equations with trivial solution. Hence all coefficients of k^i are equal to zero which yields that $\binom{n}{i}k^{n-i}[2F(s^{n-i})-\varphi(s^{n-i})F(e)-F(e)\varphi(s^{n-i})]=0$ for every $s\in R$. In particular, replacing $i=n-1$, we obtain $n[2F(s)-F(e)\varphi(s)-\varphi(s)F(e)]=0$ for all $s\in R$. Using torsion restriction on R, we obtain

$$2F(s)=\varphi(s)F(e)+F(e)\varphi(s),\ \forall s\in R. \tag{2.4}$$

Substituting $r+ke$ for r into equation (2.3), we get:

$$\begin{aligned}3\sum_{i=0}^{2n}\binom{2n}{i}F(r^{2n-i}(ke)^i) &= [\sum_{i=0}^{n}\binom{n}{i}F(r^{n-i}(ke)^i)]\varphi[\sum_{i=0}^{n}\binom{n}{i}(r^{n-i}(ke)^i)]\\ &\quad+\varphi[\sum_{i=0}^{n}\binom{n}{i}(r^{n-i}(ke)^i)]F(e)\varphi[\sum_{i=0}^{n}\binom{n}{i}(r^{n-i}(ke)^i)]\\ &\quad+\varphi[\sum_{i=0}^{n}\binom{n}{i}(r^{n-i}(ke)^i)][\sum_{i=0}^{n}\binom{n}{i}F(r^{n-i}(ke)^i)],\end{aligned}$$

$\forall r\in R, k\in\mathbb{Z}^{+}$. Reshuffling the terms of k^i for all $i=1,2,3,\ldots,2n-1$, we obtain

$$\begin{aligned}&k[3\tbinom{2n}{1}F(r^{2n-1})-\tbinom{n}{1}F(r^n)\varphi(r^{n-1})-\tbinom{n}{1}F(r^{n-1})\varphi(r^n)-\tbinom{n}{1}\varphi(r^n)F(e)\varphi(r^{n-1})\\ &-\tbinom{n}{1}\varphi(r)F(e)\varphi(r^n)-\tbinom{n}{1}\varphi(r^{n-1})F(r^n)-\tbinom{n}{1}\varphi(r^n)F(r^{n-1})]+\ldots+k^{2n-2}[3\tbinom{2n}{2n-2}F(r^2)\\ &-\tbinom{n}{n-2}F(r)\varphi(r)-\tbinom{n}{n-2}F(e)\varphi(r^2)-\tbinom{n}{n-1}\tbinom{n}{n-1}\varphi(r)F(e)\varphi(r)-\tbinom{n}{n-2}\varphi(r)F(r)\\ &-\tbinom{n}{n-2}\varphi(r^2)F(e)]+k^{2n-1}[3\tbinom{2n}{2n-1}F(r)-\tbinom{n}{n-1}F(r)-\tbinom{n}{n-1}F(e)\varphi(r)\\ &-\tbinom{n}{n-1}\varphi(r)F(e)-\tbinom{n}{n-1}F(r)-\tbinom{n}{n-1}F(e)\varphi(r)-\tbinom{n}{n-1}\varphi(r)F(e)]=0\end{aligned}$$

Applying similar arguments, we have

$$\begin{aligned}&3\tbinom{2n}{2n-2}F(r^2)-\tbinom{n}{n-2}F(r)\varphi(r)-\tbinom{n}{n-2}F(e)\varphi(r^2)\\ &-\tbinom{n}{n-1}\tbinom{n}{n-1}\varphi(r)F(e)\varphi(r)-\tbinom{n}{n-2}\varphi(r)F(r)-\tbinom{n}{n-2}\varphi(r^2)F(e)=0,\ \forall r\in R\end{aligned} \tag{2.5}$$

Replacing s by r and s by r^2 in (2.4), we find the following two equations:

$$2F(r^2)=F(e)\varphi(r^2)+\varphi(r^2)F(e),\ \forall r\in R. \tag{2.6}$$

Using (2.4), we get the following (2.7) and (2.8):

$$2\varphi(r)F(r)=\varphi(r)F(e)\varphi(r)+\varphi(r^2)F(e),\ \forall r\in R. \tag{2.7}$$

$$2\mathcal{F}(r)\varphi(r) = \varphi(r)\mathcal{F}(e)\varphi(r) + \mathcal{F}(e)\varphi(r^2),\ \forall r \in R. \tag{2.8}$$

Adding the above two equations and using (2.6), we find that

$$\varphi(r)\mathcal{F}(e)\varphi(r) = \varphi(r)\mathcal{F}(r) + \mathcal{F}(r)\varphi(r) - \mathcal{F}(r^2),\ \forall r \in R. \tag{2.9}$$

Using (2.6) and (2.9) in (2.5) and torsion restriction on R, we get $2\mathcal{F}(r^2) = \varphi(r)\mathcal{F}(r) + \mathcal{F}(r)\varphi(r)$ for every $r \in R$. Therefore, by Lemma 1.1, we reach the desired conclusion.

Next, the following example shows that the above results are not insignificant:

Example 2.1. *Define the mappings* $\mathcal{F}, \varphi$ *from a ring* $R \to R$ *by* $\mathcal{F}\left[\begin{pmatrix} r & s \\ 0 & t \end{pmatrix}\right] = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}, \varphi\left[\begin{pmatrix} r & s \\ 0 & t \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix}$, *where* $R = \left\{\begin{pmatrix} r & s \\ 0 & t \end{pmatrix} \mid r, s, t \in 2\mathbb{Z}_8\right\}$. *One can easily see that the ring* R *is not a 2-torsion free semiprime and* $\mathcal{F}$ *satisfies the algebraic identity (2.1) but* $\mathcal{F}$ *is not a centralizer. Consequently, the hypothesis of semiprime is crucial for Theorem 2.1.*

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