International Journal of Mathematics and
Computer Science, **19**(2024), no. 2, 315–318

On the Controllability for the 1D-Heat Equation with Dirichlet Boundary condition, in the Presence of a Scale-Invariant Parameter

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(Received August 2, 2023, Accepted September 17, 2023,
Published November 10, 2023)

Abstract

In this paper we study the controllability for the 1D-Heat equation with a Dirichlet boundary condition, in the presence of a scale-invariant parameter. First, we construct the scale-invariant solutions for the one-dimensional heat equation. Then we present our problem statement. We finally prove the Dirichlet boundary controllability.

1 Introduction

The controllability of partial differential equation is a very relevant area of research and has been the subject of many papers in recent years. In particular, in the context of heat equation, the one-dimensional controllability problem has been investigated in [1]. It is now well known that for the heat equation, the controllability with scale-invariant parameter is still not developed, which motivates this work.

Key words and phrases: Controllability of heat equation, Scale-invariant parameter.
AMS (MOS) Subject Classifications: 93B03, 93C20.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

In this paper, we focus on the Dirichlet boundary controllability of the 1D-Heat equation, in the presence of scale-Invariant parameter.
The rest of the paper is organized as follows:
In Section 2, we construct the scale-invariant solutions for the one-dimensional heat equation and we present our general problem formulation. In Section 3, we prove the Dirichlet boundary controllability in the presence of a scale-invariant parameter.

2 Scale-Invariant Solutions and Problem Statement

It is well-known that the one dimensional linear heat equation has a natural scaling invariance [2]. Let us denote by $y(t,x)$ an analytical solution, for any $(x,t) \in \mathbb{R}\times[0,t]$. Then, for any strictly positive real number Λ, the mapping $(t,x) \mapsto y(t,x,\Lambda) = \Lambda y(\Lambda^2 t, \Lambda x)$ is also a solution.
The exact analytical solution $y(t,x,\Lambda)$, which depends on the space variable x, the time variable t, and the scaling parameter Λ, is given by [3]:

$$y(x,t,\Lambda) = y(x,t) + \varepsilon \sum_{j=1}^{N_0} \frac{1}{\Lambda}\, y\left(\frac{x}{\Lambda^j}, \frac{t}{\Lambda^{2j}}\right), \quad \varepsilon \in \{-1,+1\},\ N_0 \in \mathbb{N}^\star.$$

One builds thus an exact solution of the one-dimensional heat equation. The dependence of this solution on the scaling parameter Λ naturally leads to a control problem governed by the 1D-Heat equation with Dirichlet boundary condition, in the presence of a scale parameter. Let $\Omega = (0,l)$. For a time $T > 0$ and for a scale-invariant parameter $\Lambda > 0$, we set $Q = \Omega\times(0,T)\times\mathbb{R}_+^\star$, $\Sigma = \partial\Omega\times(0,T)\times\mathbb{R}_+^\star$, and $K = (0,T)\times\mathbb{R}_+^\star$. We consider the following Dirichlet boundary control problem:

$$\frac{\partial y}{\partial t}(x,t,\Lambda) = \frac{\partial^2 y}{\partial x^2}(x,t,\Lambda) \qquad \text{in Q}, \tag{2.1}$$

$$y(x,0,\Lambda) = y_0(x,\Lambda) \qquad \text{in } \Omega\times\mathbb{R}_+^\star, \tag{2.2}$$

$$y(0,t,\Lambda) = 0 \qquad \text{on } \Sigma, \tag{2.3}$$

$$y(l,t,\Lambda) = u(t,\Lambda) \qquad \text{on } \Sigma, \tag{2.4}$$

where $y_0(x,\Lambda)$ is the initial temperature at the time $t = 0$, $u(t,\Lambda)$ is the control variable on the the boundary condition, in the presence of a scale-invariant parameter.

The controllability of (2.1)-(2.4) we deal with reads as follows:

Let $y_T(x,.)$ be a given function that represents a desired state we hope to reach at a time T. For each $y_0 \in L^2\left(\Omega \times \mathbb{R}_+^\star\right)$ and for any scale invariant parameter $\Lambda > 0$, find $u \in L^2(K)$ such that the corresponding solution of (2.1)-(2.4) satisfies:

$$y(x,T,\Lambda) = y_T, \text{ in } \Omega \times \mathbb{R}_+^\star \tag{2.5}$$

3 Dirichlet boundary Controllability of (2.1)-(2.4)

The objective of (2.1)-(2.4) is to find a Dirichlet control u enforced at $x = l$ satisfying 2.5. So let f be arbitrarily given in $L^2(Q)$. Consider φ as the solution of the backward heat equation:

$$\begin{cases} -\dot{\varphi} - \Delta\varphi = f & \text{in}\, Q\,, \\ \varphi(0,t,\Lambda) = 0 & \text{on}\, \Sigma\,, \\ \varphi(l,t,\Lambda) = 0 & \text{on}\, \Sigma\,, \\ \varphi(x,T,\Lambda) = 0 & \text{in}\, \Omega \times \mathbb{R}_+^\star \end{cases} \tag{3.6}$$

As recalled in [4], it is well known that, for every $f \in L^2(Q)$, there exists a unique solution φ to (3.6), with

$$\varphi \in L^2\left(0,T;H_0^1(\Omega \times \mathbb{R}_+^\star)\right) \cap C\left(0,T;L^2(\Omega \times \mathbb{R}_+^\star)\right).$$

As long as the boundary is regular enough which is assumed from now on, the normal derivative $\dfrac{\partial\varphi}{\partial x}(l,.)$ belongs to $L^2(K)$ and is subject to the stability:

$$\|\frac{\partial\varphi}{\partial x}(l,.)\|_{L^2(K)} \leq C_0\|f\|_{L^2(Q)}, \tag{3.7}$$

where C_0 denotes a positive constant.
Applying the transposition method [5] to problem (2.1)-(2.4), we come up with the following variational equation:

$$\int_Q y(x,t,\Lambda)f\,dx\,dt = -\int_K u\frac{\partial\varphi}{\partial x}(l,.)\,dt, \quad \forall f \in L^2\left(Q\right), \, u \in L^2\left(K\right) \tag{3.8}$$

The equivalence between (3.8) and problem (2.1)-(2.4) has been checked for instance in [6].
The existence and uniqueness for $, y \in L^2\left(Q\right)$ are direct consequences of (3.7) and Riesz' Theorem [7]. Consequently,

Proposition 3.1. *For any strictly positive real time T, for any scale invariant parameter $\Lambda > 0$, and for any continuous function u of $L^2(K)$, the system (2.1)-(2.4) has a unique solution $y \in L^2(Q)$ which belongs to $C\left(0, T; H^{-1}(\Omega \times \mathbb{R}_+^\star)\right)$.*

As indicated and discussed in [6] for the optimal control problem, the point is to consider a subspace of admissible Dirichlet controls that brings facilities in the computations. That should be a subspace of $L^2(K)$ that allows us to define the final observation $y(T) \in L^2\left(\Omega \times \mathbb{R}_+^\star\right)$.

Making use of Green's formula [6], we have the following result:

Corollary 3.2. *For each $y_0 \in L^2\left(\Omega \times \mathbb{R}_+^\star\right)$ and for any scale invariant parameter $\Lambda > 0$, there exists a control u, belonging to $L^2(K)$ such that the corresponding solution of (2.1)-(2.4) satisfies (2.5).*

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