On the Controllability for the 1D-Heat Equation with Dirichlet Boundary condition, in the Presence of a Scale-Invariant Parameter

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Abstract

In this paper we study the controllability for the 1D-Heat equation with a Dirichlet boundary condition, in the presence of a scale-invariant parameter. First, we construct the scale-invariant solutions for the one-dimensional heat equation. Then we present our problem statement. We finally prove the Dirichlet boundary controllability.

1 Introduction

The controllability of partial differential equation is a very relevant area of research and has been the subject of many papers in recent years. In particular, in the context of heat equation, the one-dimensional controllability problem has been investigated in [1]. It is now well known that for the heat equation, the controllability with scale-invariant parameter is still not developed, which motivates this work.

Key words and phrases: Controllability of heat equation, Scale-invariant parameter.

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In this paper, we focus on the Dirichlet boundary controllability of the 1D-Heat equation, in the presence of scale-Invariant parameter.

The rest of the paper is organized as follows:
In Section 2, we construct the scale-invariant solutions for the one-dimensional heat equation and we present our general problem formulation. In Section 3, we prove the Dirichlet boundary controllability in the presence of a scale-invariant parameter.

2 Scale-Invariant Solutions and Problem Statement

It is well-known that the one dimensional linear heat equation has a natural scaling invariance [2]. Let us denote by \( y(t, x) \) an analytical solution, for any \((x, t) \in \mathbb{R} \times [0, t] \). Then, for any strictly positive real number \( \Lambda \), the mapping \((t, x) \mapsto y(t, x, \Lambda) = \Lambda y(\Lambda^2 t, \Lambda x) \) is also a solution.

The exact analytical solution \( y(t, x, \Lambda) \), which depends on the space variable \( x \), the time variable \( t \), and the scaling parameter \( \Lambda \), is given by [3]:

\[
y(x, t, \Lambda) = y(x, t) + \varepsilon \sum_{j=1}^{N_0} \frac{1}{\Lambda} y \left( \frac{x}{\Lambda^j}, \frac{t}{\Lambda^{2j}} \right), \quad \varepsilon \in \{-1, +1\}, \quad N_0 \in \mathbb{N}^*.
\]

One builds thus an exact solution of the one-dimensional heat equation. The dependence of this solution on the scaling parameter \( \Lambda \) naturally leads to a control problem governed by the 1D-Heat equation with Dirichlet boundary condition, in the presence of a scale parameter. Let \( \Omega = (0, l) \). For a time \( T > 0 \) and for a scale-invariant parameter \( \Lambda > 0 \), we set \( Q = \Omega \times (0, T) \times \mathbb{R}_+^* \), \( \Sigma = \partial \Omega \times (0, T) \times \mathbb{R}_+^* \), and \( K = (0, T) \times \mathbb{R}_+^* \). We consider the following Dirichlet boundary control problem:

\[
\frac{\partial y}{\partial t}(x, t, \Lambda) = \frac{\partial^2 y}{\partial x^2}(x, t, \Lambda) \quad \text{in} \ Q, \quad (2.1)
\]

\[
y(x, 0, \Lambda) = y_0(x, \Lambda) \quad \text{in} \ \Omega \times \mathbb{R}_+^*, \quad (2.2)
\]

\[
y(0, t, \Lambda) = 0 \quad \text{on} \ \Sigma, \quad (2.3)
\]

\[
y(l, t, \Lambda) = u(t, \Lambda) \quad \text{on} \ \Sigma, \quad (2.4)
\]

where \( y_0(x, \Lambda) \) is the initial temperature at the time \( t = 0 \), \( u(t, \Lambda) \) is the control variable on the the boundary condition, in the presence of a scale-invariant parameter.
The controllability of (2.1)-(2.4) we deal with reads as follows:

Let \( y_{T}(x,.) \) be a given function that represents a desired state we hope to reach at a time \( T \). For each \( y_0 \in L^2(\Omega \times \mathbb{R}_+^*) \) and for any scale invariant parameter \( \Lambda > 0 \), find \( u \in L^2(K) \) such that the corresponding solution of (2.1)-(2.4) satisfies:

\[
y(x,T,\Lambda) = y_T, \quad \text{in } \Omega \times \mathbb{R}_+^* \tag{2.5}
\]

3 Dirichlet boundary Controllability of (2.1)-(2.4)

The objective of (2.1)-(2.4) is to find a Dirichlet control \( u \) enforced at \( x = l \) satisfying 2.5. So let \( f \) be arbitrarily given in \( L^2(Q) \). Consider \( \varphi \) as the solution of the backward heat equation:

\[
\begin{cases}
-\dot{\varphi} - \Delta \varphi = f & \text{in } Q, \\
\varphi(0,t,\Lambda) = 0 & \text{on } \Sigma, \\
\varphi(l,t,\Lambda) = 0 & \text{on } \Sigma, \\
\varphi(x,T,\Lambda) = 0 & \text{in } \Omega \times \mathbb{R}_+^* 
\end{cases} \tag{3.6}
\]

As recalled in [4], it is well known that, for every \( f \in L^2(Q) \), there exists a unique solution \( \varphi \) to (3.6), with

\[
\varphi \in L^2(0,T; H^1_0(\Omega \times \mathbb{R}_+^*)) \cap C(0,T; L^2(\Omega \times \mathbb{R}_+^*)) .
\]

As long as the boundary is regular enough which is assumed from now on, the normal derivative \( \frac{\partial \varphi}{\partial x}(l,.) \) belongs to \( L^2(K) \) and is subject to the stability:

\[
\| \frac{\partial \varphi}{\partial x}(l,.) \|_{L^2(K)} \leq C_0 \| f \|_{L^2(Q)}, \tag{3.7}
\]

where \( C_0 \) denotes a positive constant.

Applying the transposition method [5] to problem (2.1)-(2.4), we come up with the following variational equation:

\[
\int_Q y(x,t,\Lambda) f \, dx \, dt = - \int_K u \frac{\partial \varphi}{\partial x}(l,.) \, dt, \quad \forall f \in L^2(Q), \; u \in L^2(K) \quad \tag{3.8}
\]

The equivalence between (3.8) and problem (2.1)-(2.4) has been checked for instance in [6].

The existence and uniqueness for \( y \in L^2(Q) \) are direct consequences of (3.7) and Riesz' Theorem [7]. Consequently,
Proposition 3.1. For any strictly positive real time $T$, for any scale invariant parameter $\Lambda > 0$, and for any continuous function $u$ of $L^2(K)$, the system (2.1)-(2.4) has a unique solution $y \in L^2(Q)$ which belongs to $C\left(0, T; H^{-1}(\Omega \times \mathbb{R}^*_+)\right)$.

As indicated and discussed in [6] for the optimal control problem, the point is to consider a subspace of admissible Dirichlet controls that brings facilities in the computations. That should be a subspace of $L^2(K)$ that allows us to define the final observation $y(T) \in L^2(\Omega \times \mathbb{R}^*_+)$.

Making use of Green’s formula [6], we have the following result:

Corollary 3.2. For each $y_0 \in L^2(\Omega \times \mathbb{R}^*_+)$ and for any scale invariant parameter $\Lambda > 0$, there exists a control $u$, belonging to $L^2(K)$ such that the corresponding solution of (2.1)-(2.4) satisfies (2.5).

References


