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A Product of Tree Languages

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Abstract

Let $W_\tau(X)$ be the set of all terms of type τ. Any element of the power set $P(W_\tau(X))$ is called a tree language. In this paper, we define a new binary associative operation $\cdot_{ij}$ on $P(W_\tau(X))$ and so a new semigroup is obtained. We study the algebraic structures of such a semigroup including idempotent elements, regular elements, and Green's relations.

1 Introduction

Let $\tau := (n_i)_{i\in I}$ be a type with n_i-ary operation symbols f_i. For an integer $n \geq 1$, let $X_n := \{x_1, \ldots, x_n\}$ be the set of variables $x_1, \ldots, x_n$ and $X := \{x_1, x_2, \ldots\}$. The *n-ary terms* [3] of type τ are defined as follows:

(i) Every variable $x_j \in X_n$ is an n-ary term for $j = 1, ..., n$.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term.

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We let $W_\tau(X_n)$ denote the set of all n-ary terms of type τ which is the smallest set containing all variables in X_n and closed under finite application of (ii). For a countably infinite set $X = \{x_1, x_2, ...\}$, $W_\tau(X)$ denotes the set of all terms of type τ, where $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ ([4]).

Sets of terms of type τ are called *tree languages*. The *tree language product* is an important operation defined on sets of tree languages which maps recognizable tree languages to recognizable ones. This tree language product can be described by the superposition of sets of terms. In the theory of tree languages, the product of languages is called the *z-product* [4]. In [2], Denecke and Sarasit studied properties of the arising semigroups and their subsemigroups. They were especially interested in idempotent and regular elements, Green's relations $\mathcal{L}$ and $\mathcal{R}$. Based on the superposition operation, we define a new binary associative operation on the set of all tree languages. The purpose of this work is to investigate some important properties of the semigroup of the set of all tree languages of type τ together with a new product of such tree languages.

We denote by $W_\tau(X)$ the set of all terms of type $\tau = (n_i)_{i \in I}$. Any element of the power set $P(W_\tau(X))$ is called a *tree language*.

In [1], on the set $P(W_\tau(X))$, an $(n+1)$-ary superposition operation

$$\hat{S}_g^n : P(W_\tau(X))^{n+1} \to P(W_\tau(X))$$

is inductively defined by the following steps:

Let $n \in \mathbb{N}^+$ $(:= \mathbb{N} \setminus \{0\})$ be a natural number and let $B, B_1, ..., B_n \in P(W_\tau(X))$ such that $B, B_1, ..., B_n$ are non-empty.

(1) If $B = \{x_i\}$ for $1 \leq i \leq n$, then $\hat{S}_g^n(\{x_i\}, B_1, ..., B_n) := B_i$, and if $B = \{x_i\}$ for $n < i$, then $\hat{S}_g^n(\{x_i\}, B_1, ..., B_n) := \{x_i\}$.

(2) If $B = \{f_i(t_1, ..., t_{n_i})\}$ and if we assume that $\hat{S}_g^n(\{t_j\}, B_1, ..., B_n)$ for $1 \leq j \leq n_i$ are already defined, then $\hat{S}_g^n(\{f_i(t_1, ..., t_{n_i})\}, B_1, ..., B_n) := \{f_i(r_1, ..., r_{n_i}) \mid r_j \in \hat{S}_g^n(\{t_j\}, B_1, ..., B_n), 1 \leq j \leq n_i\}$.

(3) If B is an arbitrary non-empty subset of $W_\tau(X)$, then we define $\hat{S}_g^n(B, B_1, ..., B_n) := \bigcup_{b \in B} \hat{S}_g^n(\{b\}, B_1, ..., B_n)$.

If one of $B, B_1, ..., B_n$ is empty, then we define $\hat{S}_g^n(B, B_1, ..., B_n) = \emptyset$.

The operation $\hat{S}^n_g$ satisfies the following equation $(Cg1)$, which is called the *superassociative law*:

$$\tilde{S}^n(T, \tilde{S}^n(F_1, T_1, ..., T_n), ..., \tilde{S}^n(F_n, T_1, ..., T_n)) \approx \tilde{S}^n(\tilde{S}^n(T, F_1, ..., F_n), T_1, ..., T_n)$$

Here $\tilde{S}^n$ is an operation symbol corresponding to the operation $\hat{S}^n_g$.

Let $i \leq n$. Denecke and Sarasit [2] defined a binary operation $\cdot_{x_i}$ by

$$B_1 \cdot_{x_i} B_2 := \hat{S}^n_g(B_1, \{x_1\}, ..., \{x_{i-1}\}, B_2, \{x_{i+1}\}, ..., \{x_n\})$$

for all $B_1, B_2 \subseteq W_\tau(X)$. Because of (Cg1), the operation $\cdot_{x_i}$ is associative and so $(P(W_\tau(X)); \cdot_{x_i})$ is a semigroup. Since $A \cdot_{x_i} \{x_i\} = A = \{x_i\} \cdot_{x_i} A$ for all $A \in P(W_\tau(X))$, the set $\{x_i\}$ is an identity element with respect to the multiplication $\cdot_{x_i}$ and the algebra $(P(W_\tau(X)); \cdot_{x_i}, \{x_i\})$ is a monoid.

2 The semigroup $(P(W_\tau(X)); \cdot_{ij})$

Using the operation $\hat{S}^n_g : P(W_\tau(X))^{n+1} \to P(W_\tau(X))$ for every $n \geq 1$ and $1 \leq i \leq j \leq n$, we define a binary operation $\cdot_{ij}$ as follows:

$$A \cdot_{ij} B := \hat{S}^n_g(A, \{x_1\}, ..., \{x_{i-1}\}, B, \{x_{i+1}\}, ..., \{x_{j-1}\}, B, \{x_{j+1}\}, ..., \{x_n\})$$

for all $A, B \in P(W_\tau(X))$.

Example 2.1. *Let $X = \{x_1, x_2, x_3\}$ and $\tau = (3, 1)$ with a ternary operation f and a unary operation g. Let $i = 1, j = 3, A = \{g(f(x_1, x_2, x_3))\},$ and $B = \{g(x_2), f(x_2, x_3, x_1)\}$. Then*

$$\begin{aligned} A \cdot_{13} B &= \hat{S}^3_g(A, B, \{x_2\}, B) \\ &= \{g(f(g(x_2), x_2, g(x_2))), g(f(g(x_2), x_2, f(x_2, x_3, x_1))), \\ &\quad g(f(f(x_2, x_3, x_1), x_2, g(x_2))), g(f(f(x_2, x_3, x_1), x_2, f(x_2, x_3, x_1)))\}. \end{aligned}$$

Since $\hat{S}^n_g$ satisfies (Cg1), $\cdot_{ij}$ is associative and so $(P(W_\tau(X)); \cdot_{ij})$ is a semigroup. Next, we show that the semigroup $(P(W_\tau(X)); \cdot_{ij})$ does not have an identity. If B is an identity, then B is a variable. Let $B \in P(W_\tau(X))$. Clearly, if $B = \{x_s\}, s \neq i, s \neq j$, then B is not an identity. If $B = \{x_i\}$ and there is $A = \{x_j\}$ for $i \neq j$, then $A \cdot_{ij} B = \{x_i\} \neq \{x_j\} = A$. If $B = \{x_j\}$ and there is $A = \{x_i\}$ for $i \neq j$, then $A \cdot_{ij} B = \{x_j\} \neq \{x_i\} = A$. Therefore, $(P(W_\tau(X)); \cdot_{ij})$ does not have an identity.

Let $Var(A)$ be the set of all variables occurring in some terms of A. The following lemmas show the properties of $\cdot_{ij}$.

Lemma 2.2. *Let* $A, B \in P(W_\tau(X))$ *and let* $1 \leq i \leq j \leq n$. *If* $x_i, x_j \notin Var(A)$ *and* $B \neq \emptyset$, *then* $A \cdot_{ij} B = A$.

Proof. Assume that $x_i, x_j \notin Var(A)$ and $B \neq \emptyset$. If $A = \emptyset$, then $A \cdot_{ij} B = A$. Next, assume that $A \neq \emptyset$. We show that for all terms $t \in W_\tau(X)$ such that $x_i, x_j \notin Var(\{t\})$, we have $\{t\} \cdot_{ij} B = \{t\}$. We will proceed by induction on the complexity of the term t. If $t = x_s$ for some $1 \leq s \leq n$ and $s \neq i, s \neq j$, then $\{x_s\} \cdot_{ij} B = \{x_s\}$. If $t = f_i(t_1, ..., t_{n_i})$, then $x_i, x_j \notin Var(\{t_k\})$ for all $1 \leq k \leq n_i$. We assume that $\{t_k\} \cdot_{ij} B = \{t_k\}$ for all $1 \leq k \leq n_i$. Then

$$\begin{aligned}
\{t\} \cdot_{ij} B &= \hat{S}^n_g(\{f_i(t_1, \ldots, t_{n_i})\}, \{x_1\}, \ldots, \{x_{i-1}\}, B, \{x_{i+1}\}, \ldots, \{x_{j-1}\}, B, \\
&\quad \{x_{j+1}\}, \ldots, \{x_n\}) \\
&= \{f_i(r_1, \ldots, r_{n_i}) \mid r_k \in \hat{S}^n_g(\{t_k\}, \{x_1\}, \ldots, \{x_{i-1}\}, B, \{x_{i+1}\}, \ldots, \\
&\quad \{x_{j-1}\}, B, \{x_{j+1}\}, \ldots, \{x_n\}), 1 \leq k \leq n_i\} \\
&= \{f_i(r_1, \ldots, r_{n_i}) \mid r_k \in \{t_k\}, 1 \leq k \leq n_i\} \\
&= \{f_i(t_1, \ldots, t_{n_i})\} = \{t\}.
\end{aligned}$$

If $x_i, x_j \notin Var(A)$, then $x_i, x_j \notin Var(\{t\})$ for all $t \in A$. Then we have $A \cdot_{ij} B = (\bigcup_{a \in A} \{a\}) \cdot_{ij} B = \bigcup_{a \in A} (\{a\} \cdot_{ij} B) = \bigcup_{a \in A} \{a\} = A$. $\square$

We note that if $\{f_i(t_1, \ldots, t_{n_i})\}, B \in P(W_\tau(X))$, $i, j \in \{1, \ldots, n\}$ and $i \leq j$, then for each $1 \leq k \leq n_i$, $Var(\{t_k\} \cdot_{ij} B) \subseteq Var(\{f_i(t_1, \ldots, t_{n_i})\} \cdot_{ij} B)$.

Lemma 2.3. *Let* $A, B \in P(W_\tau(X))$ *and let* $i, j \in \{1, \ldots, n\}$ *and* $i \leq j$. *If* $x_i \in Var(A)$ *or* $x_j \in Var(A)$, *then* $Var(B) \subseteq Var(A \cdot_{ij} B)$.

Lemma 2.4. *Let* $A, B \in P(W_\tau(X))$ *and let* $i, j \in \{1, \ldots, n\}$ *and* $i \leq j$. *Then* $x_i, x_j \notin Var(A \cdot_{ij} B)$ *if and only if* $x_i, x_j \notin Var(A)$ *or* $x_i, x_j \notin Var(B)$.

Proof. Assume that $x_i, x_j \notin Var(A \cdot_{ij} B)$. Suppose that $x_i \in Var(A)$ or $x_j \in Var(A)$. By Lemma 2.3, $x_i, x_j \notin Var(A \cdot_{ij} B) \supseteq Var(B)$. Conversely, assume that $x_i, x_j \notin Var(A)$ or $x_i, x_j \notin Var(B)$. If $A = \emptyset$ or $B = \emptyset$, then $x_i, x_j \notin \emptyset = Var(A \cdot_{ij} B)$. Next, we consider $A \neq \emptyset$ and $B \neq \emptyset$. If $x_i, x_j \notin Var(A)$, then by Lemma 2.2, $A \cdot_{ij} B = A$, and so $x_i, x_j \notin Var(A \cdot_{ij} B) = Var(A)$. If $x_i, x_j \notin Var(B)$, then we will proceed by induction on the complexity of the set of terms A. If $A = \{x_i\}$ or $A = \{x_j\}$, then $A \cdot_{ij} B = B$, and so $x_i, x_j \notin Var(A \cdot_{ij} B)$. If $A = \{x_s\}$ where $s \neq i$ and $s \neq j$, then $A \cdot_{ij} B = A = \{x_s\}$, and so $x_i, x_j \notin Var(A \cdot_{ij} B)$. If $A = \{f_i(t_1, \ldots, t_{n_i})\}$ and assuming that $x_i, x_j \notin Var(\{t_k\} \cdot_{ij} B)$ for all $1 \leq k \leq n_i$, then $x_i, x_j \notin \bigcup_{k=1}^{n_i} Var(\{t_k\} \cdot_{ij} B) = Var(A \cdot_{ij} B)$. If A is an arbitrary non-empty set, then for all $a \in A$ such that $x_i, x_j \notin Var(\{a\} \cdot_{ij} B)$, and so $x_i, x_j \notin \bigcup_{a \in A} Var(\{a\} \cdot_{ij} B) = Var(A \cdot_{ij} B)$. □

Lemma 2.5. *Let $A, B \in P(W_\tau(X))$ and let $i, j \in \{1, \ldots, n\}$ and $i \leq j$. Then $x_i, x_j \notin A \cdot_{ij} B$ if and only if $x_i, x_j \notin A$ or $x_i, x_j \notin B$.*

Proof. Assume that $x_i, x_j \notin A \cdot_{ij} B$. Suppose that $x_i \in A$ or $x_j \in A$. Then $B = \{x_i\} \cdot_{ij} B \subseteq \bigcup_{a \in A} \{a\} \cdot_{ij} B = A \cdot_{ij} B$ or $B = \{x_j\} \cdot_{ij} B \subseteq \bigcup_{a \in A} \{a\} \cdot_{ij} B = A \cdot_{ij} B$, and so $x_i, x_j \notin A \cdot_{ij} B \supseteq B$. Conversely, assume that $x_i, x_j \notin A$ or $x_i, x_j \notin B$. If $A = \emptyset$ or $B = \emptyset$, then $x_i, x_j \notin \emptyset = A \cdot_{ij} B$. Next, consider $A \neq \emptyset$ and $B \neq \emptyset$.

1. $x_i, x_j \notin A$.
   If $A = \{x_s\}$ where $s \neq i, s \neq j$, then $A \cdot_{ij} B = \{x_s\}$, so $x_i, x_j \notin A \cdot_{ij} B$.
   If $A = \{f_i(t_1, \ldots, t_{n_i})\}$, then $x_i, x_j \notin A \cdot_{ij} B$ is clear.
   If A is a non-empty arbitrary set, then for all $a \in A$, we have $x_i, x_j \notin \{a\} \cdot_{ij} B$, and so $x_i, x_j \notin \bigcup_{a \in A} \{a\} \cdot_{ij} B = A \cdot_{ij} B$.
2. $x_i, x_j \notin B$.
   If $A = \{x_s\}$ where $s = i$ or $s = j$, then $A \cdot_{ij} B = B$, and so $x_i, x_j \notin A \cdot_{ij} B$.
   If $A = \{x_s\}$ where $s \neq i, s \neq j$, then $A \cdot_{ij} B = \{x_s\}$, and so $x_i, x_j \notin A \cdot_{ij} B$.
   If $A = \{f_i(t_1, \ldots, t_{n_i})\}$, then $x_i, x_j \notin A \cdot_{ij} B$ is clear.
   If A is a non-empty arbitrary set, then for all $a \in A$, we have $x_i, x_j \notin \{a\} \cdot_{ij} B$, and so $x_i, x_j \notin \bigcup_{a \in A} \{a\} \cdot_{ij} B = A \cdot_{ij} B$. □

Lemma 2.6. *Let $A, B \in P(W_\tau(X))$ and let $i, j \in \{1, \ldots, n\}$ and $i \leq j$. If $x_i \in A \cdot_{ij} B$ or $x_j \in A \cdot_{ij} B$, then $B \subseteq A \cdot_{ij} B$.*

Proof. By Lemma 2.5, ($x_i \in A$ or $x_j \in A$) and ($x_i \in B$ or $x_j \in B$). Then

$$\begin{aligned} B &= \hat{S}^n_g(\{x_k\}, \{x_1\}, \ldots, \{x_{i-1}\}, B, \{x_{i+1}\}, \ldots, \{x_{j-1}\}, B, \{x_{j+1}\}, \ldots, \{x_n\}) \\ &\subseteq \hat{S}^n_g(A, \{x_1\}, \ldots, \{x_{i-1}\}, B, \{x_{i+1}\}, \ldots, \{x_{j-1}\}, B, \{x_{j+1}\}, \ldots, \{x_n\}) \\ &= A \cdot_{ij} B \text{ where } k = i \text{ or } k = j. \end{aligned}$$

Therefore, $B \subseteq A \cdot_{ij} B$. □

Lemma 2.7. *Let $A, B \in P(W_\tau(X))$ and let $i, j, k \in \{1, \ldots, n\}$ and $i \leq j$. If $x_k \in A \cdot_{ij} B$ and $x_k \notin A$, then $B \subseteq A \cdot_{ij} B$ and $x_k \in B$.*

Proof. Assume that $x_k \in A \cdot_{ij} B$ and $x_k \notin A$. Then there exists $a \in A$ such that $x_k \in \{a\} \cdot_{ij} B$. Suppose that $a \neq x_i$ and $a \neq x_j$. Then $x_k \in \{a\} \cdot_{ij} B = \{a\}$, and so $x_k \in \{a\} \subseteq A$, it is a contradiction. Therefore, $a = x_i$ or $a = x_j$. Then $x_k \in \{a\} \cdot_{ij} B = B$, and so $x_k \in B$. By Lemma 2.6, $B \subseteq A \cdot_{ij} B$. □

3 Idempotent and Regular Elements

An element of the semigroup $(P(W_\tau(X)); \cdot_{ij})$ is called *idempotent* if $A \cdot_{ij} A = A$ and called *regular* if $A = A \cdot_{ij} B \cdot_{ij} A$ for some $B \in P(W_\tau(X))$. Let $t \in W_\tau(X)$. The number of operation symbols occurring in t is denoted by $op(t)$. Let $A \in P(W_\tau(X))$. We define the following sets:

$$\begin{aligned} A^{'} &:= \{a | a \in A \text{ and } x_i \in Var(\{a\}) \text{ or } x_j \in Var(\{a\})\}, \\ A^{''} &:= \{a | a \in A \text{ and } x_i, x_j \notin Var(\{a\})\}, \\ A_r &:= \{a | a \in A \text{ and } op(a) = r\}. \end{aligned}$$

We observe that $A = A^{'} \cup A^{''}$.

In the case $x_i, x_j \notin Var(A)$, $A \cdot_{ij} A = A$ by Lemma 2.2, i.e., A is idempotent. For the case $x_i \in Var(A)$ or $x_j \in Var(A)$, we show that if A is idempotent, then $x_i \in A$ or $x_j \in A$. The following lemma is a more general result.

Lemma 3.1. *Let $A, B \in P(W_\tau(X))$ and $x_i \in Var(A)$ or $x_j \in Var(A)$. If $A = B \cdot_{ij} A$ or $A = A \cdot_{ij} B$, then $x_i \in B$ or $x_j \in B$.*

Proof. Assume that $A = B \cdot_{ij} A$. If $x_i, x_j \notin Var(B)$, then $A = B \cdot_{ij} A = B$, and so $x_i, x_j \notin Var(A)$, a contradiction. Therefore, $x_i \in Var(B)$ or $x_j \in Var(B)$ which means $B^{'} \neq \emptyset$. Suppose that $x_i, x_j \notin B$. Then $op(b) \geq 1$ for all $b \in B^{'}$. Since $x_i \in Var(A)$ or $x_j \in Var(A)$, $A^{'} \neq \emptyset$. Let s be the least natural number such that $A^{'}_s \neq \emptyset$. Consider $h \in A^{'}$; that is, $h \in A$ and $x_i \in Var(\{h\})$ or $x_j \in Var(\{h\})$. Because of $A = B \cdot_{ij} A$ and $B = B^{'} \cup B^{''}$, we get $A = B \cdot_{ij} A = (B^{'} \cup B^{''}) \cdot_{ij} A = (B^{'} \cdot_{ij} A) \cup (B^{''} \cdot_{ij} A)$. Since $B^{''} \cdot_{ij} A = B^{''}$, and $x_i \in Var(\{h\})$ or $x_j \in Var(\{h\})$, we have $h \in B^{'} \cdot_{ij} A$. So, $op(h) \geq 1 + s > s$ and $h \notin A^{'}_s$ for all $h \in A^{'}$, a contradiction. Then $x_i \in B$ or $x_j \in B$. Assume that $A = A \cdot_{ij} B$ and $x_i, x_j \notin B$. Then $op(b) \geq 1$ for all $b \in B^{'}$. Since $x_i \in Var(A)$ or $x_j \in Var(A)$, $A^{'} \neq \emptyset$. Let s be the least natural number such that $A^{'}_s \neq \emptyset$. Consider $h \in A^{'}$. Since $A = A^{'} \cup A^{''}$, $A = A \cdot_{ij} B = (A^{'} \cup A^{''}) \cdot_{ij} B = (A^{'} \cdot_{ij} B) \cup (A^{''} \cdot_{ij} B)$, and so $h \in A^{'} \cdot_{ij} B$. Since $B = B^{'} \cup B^{''}$ and $h \notin A^{'} \cdot_{ij} B^{''}$, $h \in A^{'} \cdot_{ij} B^{'}$. So, $op(h) \geq s + 1 > s$, and then $h \notin A^{'}_s$ for all $h \in A^{'}$, a contradiction. Then $x_i \in B$ or $x_j \in B$. □

Corollary 3.2. *Let $A \in P(W_\tau(X))$ with $x_i \in Var(A)$ or $x_j \in Var(A)$. If A is an idempotent element of $(P(W_\tau(X)); \cdot_{ij})$, then $x_i \in A$ or $x_j \in A$.*

Proof. Assume that A is an idempotent element of $(P(W_\tau(X)); \cdot_{ij})$. Then $A = A \cdot_{ij} A$. By Lemma 3.1, we get that $x_i \in A$ or $x_j \in A$. □

Lemma 3.3. *Let $A, B \in P(W_\tau(X))$ and $x_i \in Var(A)$ or $x_j \in Var(A)$. If $A = A \cdot_{ij} B$ and if there is a natural number $s \geq 1$ such that $B_s^{'} \neq \emptyset$, then A is infinite.*

Proof. Assume that A is finite. Then $A^{'}$ is also finite. Let $t \in A'$ such that t has a maximal number of occurrences of operation symbols. Then with $b \in B_s^{'}$, we have $op(b) = s \geq 1$. Consider

$$\begin{aligned} h \ &\in \hat{S}_g^n(\{t\}, \{x_1\}, ..., \{x_{i-1}\}, \{b\}, \{x_{i+1}\}, ..., \{x_{j-1}\}, \{b\}, \{x_{j+1}\}, ..., \{x_n\}) \\ &\subseteq \hat{S}_g^n(\{t\}, \{x_1\}, ..., \{x_{i-1}\}, B, \{x_{i+1}\}, ..., \{x_{j-1}\}, B, \{x_{j+1}\}, ..., \{x_n\}) \\ &\subseteq \hat{S}_g^n(A, \{x_1\}, ..., \{x_{i-1}\}, B, \{x_{i+1}\}, ..., \{x_{j-1}\}, B, \{x_{j+1}\}, ..., \{x_n\}) \\ &= A \cdot_{ij} B = A. \end{aligned}$$

Then $op(h) \geq op(t) + 1 > op(t)$ and $x_i \in Var(\{h\})$ or $x_j \in Var(\{h\})$. Hence, $h \in A^{'}$ and $op(h) > op(t)$ is a contradiction. Therefore, A is infinite. □

Corollary 3.4. *Let A be idempotent in $(P(W_\tau(X)); \cdot_{ij})$. If $A_s^{'} \neq \emptyset$ for some $s \geq 1$, then A is infinite.*

Theorem 3.5. *The set A is idempotent in $(P(W_\tau(X)); \cdot_{ij})$ if and only if it is regular.*

Proof. It is clear that every idempotent element of $P(W_\tau(X))$ is a regular element. Conversely, let A be a regular element of $P(W_\tau(X))$. Then there exists $B \in P(W_\tau(X))$ such that $A = A \cdot_{ij} B \cdot_{ij} A$. If $x_i, x_j \notin Var(A)$, then A is idempotent. If $x_i \in Var(A)$ or $x_j \in Var(A)$, then by Lemma 3.1, $x_i \in B \cdot_{ij} A$ or $x_j \in B \cdot_{ij} A$. By Lemma 2.6, we have $A \subseteq B \cdot_{ij} A \subseteq A \cdot_{ij} (B \cdot_{ij} A) = A$. Then $A = B \cdot_{ij} A$, and so $A = A \cdot_{ij} A$. Therefore, A is idempotent. □

Lemma 3.6. *Let $A \in P(W_\tau(X))$ be a regular (idempotent) element of the semigroup $(P(W_\tau(X)); \cdot_{ij})$ with $x_i \in Var(A)$ or $x_j \in Var(A)$. Then for all $\emptyset \neq B \subseteq A$ we have $x_i \in B$ or $x_j \in B$ if and only if $A = A \cdot_{ij} B \cdot_{ij} A$.*

Proof. Since A is idempotent and $x_i \in Var(A)$ or $x_j \in Var(A)$, $x_i \in A$ or $x_j \in A$ by Corollary 3.2. Let $\emptyset \neq B \subseteq A$. Assume that $x_i \in B$ or $x_j \in B$. By Lemma 2.6, $A \subseteq B \cdot_{ij} A \subseteq A \cdot_{ij} B \cdot_{ij} A \subseteq A \cdot_{ij} A \cdot_{ij} A = A \cdot_{ij} A = A$, and so $A = A \cdot_{ij} B \cdot_{ij} A$. Conversely, assume that $A = A \cdot_{ij} B \cdot_{ij} A$. If $x_i, x_j \notin B$, then by Lemma 2.5, $x_i, x_j \notin B \cdot_{ij} A$, and so $x_i, x_j \notin A \cdot_{ij} B \cdot_{ij} A = A$ is a contradiction. Thus, $x_i \in B$ or $x_j \in B$. □

4 Green's relations on $P(W_\tau(X))$

In this section, we characterize all Green's relations for the semigroup $(P(W_\tau(X)); \cdot_{ij})$. First, we recall the definition of all Green's relations.

Let $A, B \in P(W_\tau(X))$. Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ are defined as follows:

(i) $A\mathcal{L}B$ if and only if there are $C, D \in P(W_\tau(X))$ such that $C \cdot_{ij} A = B$ and $D \cdot_{ij} B = A$.

(ii) $A\mathcal{R}B$ if and only if there are $E, F \in P(W_\tau(X))$ such that $A \cdot_{ij} E = B$ and $B \cdot_{ij} F = A$.

(iii) $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

(iv) $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.

(v) $A\mathcal{J}B$ if and only if there are $C, D, E, F \in P(W_\tau(X))$ such that $C \cdot_{ij} A \cdot_{ij} D = B$ and $E \cdot_{ij} B \cdot_{ij} F = A$.

First, we characterize Green's relation $\mathcal{L}$ and $\mathcal{R}$ for our semigroup. We consider the cases $A\mathcal{L}B$ and $A\mathcal{R}B$ for $x_i, x_j \notin Var(A)$ and $x_i \in Var(A)$ or $x_j \in Var(A)$.

Theorem 4.1. *Let $A, B \in P(W_\tau(X))$.*

(i) If $x_i, x_j \notin Var(A)$, then $A\mathcal{L}B$ if and only if $x_i, x_j \notin Var(B)$.

(ii) If $x_i \in Var(A)$ or $x_j \in Var(A)$, then $A\mathcal{L}B$ if and only if $A = B$.

Proof. (i) Assume that $A\mathcal{L}B$. Then there are $C, D \in P(W_\tau(X))$ such that $A = C \cdot_{ij} B$ and $B = D \cdot_{ij} A$. Since $x_i, x_j \notin Var(A)$ and $B = D \cdot_{ij} A$, by Lemma 2.4, $x_i, x_j \notin Var(D \cdot_{ij} A) = Var(B)$. Conversely, assume that $x_i, x_j \notin Var(B)$. Since $x_i, x_j \notin Var(A)$, by Lemma 2.2, we have $B \cdot_{ij} A = B$ and $A \cdot_{ij} B = A$. Therefore, $A\mathcal{L}B$.

(ii) Assume that $A\mathcal{L}B$. Then there are $C, D \in P(W_\tau(X))$ such that $A = C \cdot_{ij} B$ and $B = D \cdot_{ij} A$. Thus $A = C \cdot_{ij} D \cdot_{ij} A$ and $B = D \cdot_{ij} C \cdot_{ij} B$. By Lemma 3.1, we have $x_i \in C \cdot_{ij} D$ or $x_j \in C \cdot_{ij} D$. By Lemma 2.5, we have x_i or $x_j \in C$ and x_i or $x_j \in D$. By Lemma 2.5, $x_i \in C \cdot_{ij} D$ or $x_j \in C \cdot_{ij} D$. By Lemma 2.6, $D \subseteq C \cdot_{ij} D$, and so $B = D \cdot_{ij} A \subseteq C \cdot_{ij} D \cdot_{ij} A = A$. Similarly, since $x_i \in C$ or $x_j \in C$, by Lemma 2.5, $x_i \in D \cdot_{ij} C$ or $x_j \in D \cdot_{ij} C$. By Lemma 2.6, $C \subseteq D \cdot_{ij} C$. Then $A = C \cdot_{ij} B \subseteq D \cdot_{ij} C \cdot_{ij} B = B$. Therefore, $A = B$. The converse is clear. $\square$

Theorem 4.2. *Let $A, B \in P(W_\tau(X))$.*

(i) If $x_i, x_j \notin Var(A)$, then $A\mathcal{R}B$ if and only if $A = B$.

(ii) If $x_i \in Var(A)$ or $x_j \in Var(A)$ and $A\mathcal{R}B$, then $x_i \in Var(B)$ or $x_j \in Var(B)$, and $\{a \in A \mid x_i, x_j \notin Var(\{a\})\} = \{b \in B \mid x_i, x_j \notin Var(\{b\})\}$.

Proof. (i) Assume that $A\mathcal{R}B$. Then there are $U, V \in P(W_\tau(X))$ such that $A = B \cdot_{ij} U$ and $B = A \cdot_{ij} V$. By Lemma 2.2 and $x_i, x_j \notin Var(A)$, we have $B = A \cdot_{ij} V = A$. The converse is clear.

(ii) Assume that $A\mathcal{R}B$. Then there are $U, V \in P(W_\tau(X))$ such that $A = B \cdot_{ij} U$ and $B = A \cdot_{ij} V$. Since $x_i \in Var(A)$ or $x_j \in Var(A)$, $x_i \in Var(B \cdot_{ij} U)$ or $x_j \in Var(B \cdot_{ij} U)$. By Lemma 2.4, we have $x_i \in Var(B)$ or $x_j \in Var(B)$ and $x_i \in Var(U)$ or $x_j \in Var(U)$. If $a \in A$ and $x_i, x_j \notin Var(\{a\})$, then $a \in \{a\} \cdot_{ij} V \subseteq A \cdot_{ij} V = B$. Similarly, if $b \in B$ and $x_i, x_j \notin Var(\{b\})$, then $b \in \{b\} \cdot_{ij} U \subseteq B \cdot_{ij} U = A$. □

Next, we consider the characterizations of the other three remaining relations $\mathcal{H}, \mathcal{D}$ and $\mathcal{J}$.

Theorem 4.3. *For $(P(W_\tau(X)); \cdot_{ij})$, the characterizations of $\mathcal{H}$ and $\mathcal{D}$ are*

(i) $\mathcal{H} = \mathcal{R}$,

(ii) $\mathcal{D} = \mathcal{L}$.

Since $\{x_1\} \cdot_{12} \{x_2\} \cdot_{12} \{x_1\} = \{x_1\}$ and $\{x_2\} \cdot_{12} \{x_1\} \cdot_{12} \{x_2\} = \{x_2\}$, by Theorem 4.1, $\mathcal{J} \nsubseteq \mathcal{L}$ and so, $\mathcal{L}$ is a proper subset of $\mathcal{J}$.

Theorem 4.4. *Let $A, B \in P(W_\tau(X))$.*

(i) If $x_i, x_j \notin Var(A)$, then $A\mathcal{J}B$ if and only if $A\mathcal{L}B$.

(ii) If $x_i \in Var(A)$ or $x_j \in Var(A)$ and $A\mathcal{J}B$, then $x_i \in Var(B)$ or $x_j \in Var(B)$, and $\{a \in A \mid x_i, x_j \notin Var(\{a\})\} = \{b \in B \mid x_i, x_j \notin Var(\{b\})\}$.

Proof. Let $A, B \in P(W_\tau(X))$.

(i) Let $x_i, x_j \notin Var(A)$. Assume that $A\mathcal{L}B$. Then there exist $U, V \in P(W_\tau(X))$ such that $A = U \cdot_{ij} B$ and $B = V \cdot_{ij} A$. By Theorem 4.1, we have $x_i, x_j \notin Var(B)$. So, $A = U \cdot_{ij} B \cdot_{ij} C$ and $B = V \cdot_{ij} A \cdot_{ij} D$ for all $C, D \in P(W_\tau(X))$. Therefore, $A\mathcal{J}B$. The converse is clear.

(ii) Let $x_i \in Var(A)$ or $x_j \in Var(A)$. Assume that $A\mathcal{J}B$. Then there exist $U, V, S, T \in P(W_\tau(X))$ such that $A = U \cdot_{ij} B \cdot_{ij} V$ and $B = S \cdot_{ij} A \cdot_{ij} T$. Since $x_i \in Var(A)$ or $x_j \in Var(A)$, $x_i \in Var(U \cdot_{ij} B \cdot_{ij} V)$ or $x_j \in Var(U \cdot_{ij} B \cdot_{ij} V)$. By Lemma 2.4, we have $x_i \in Var(U)$ or $x_j \in Var(U)$ and $x_i \in Var(B)$ or $x_j \in Var(B)$ and $x_i \in Var(V)$ or $x_j \in Var(V)$. So, $x_i \in Var(S)$ or $x_j \in Var(S)$ and $x_i \in Var(T)$ or $x_j \in Var(T)$. If $a \in A$ and $x_i, x_j \notin Var(\{a\})$, then $a \in \{a\} \cdot_{ij} T \subseteq A \cdot_{ij} T \subseteq S \cdot_{ij} A \cdot_{ij} T = B$ by Lemma 2.4 and 2.6. Similarly, if $b \in B$ and $x_i, x_j \notin Var(\{b\})$, then $b \in \{b\} \cdot_{ij} V \subseteq B \cdot_{ij} V \subseteq U \cdot_{ij} B \cdot_{ij} V = A$. □

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