International Journal of Mathematics and
Computer Science, **19**(2024), no. 2, 333–343

Generalization of Horner Division Method

Amara Chandoul[1], **Alanod M. Sibih**[2]

[1]Higher Institute of Informatics and Multimedia of Sfax
Sfax University
Sfax, Tunisia

[2]Department of Mathematics
Jamoum University College
Umm Al-Qura University
Holy Makkah 21955, Saudi Arabia

email: amarachandoul@yahoo.fr, amsibih@uqu.edu.sa

(Received August 9, 2023, Accepted September 23, 2023,
Published November 10, 2023)

Abstract

In this paper, we generalize the Horner Division method (dividing a polynomial by a monomial) to dividing a polynomial by another polynomial of any degree.

1 Introduction

Consider a polynomial of degree n over $\mathbb{R}$:

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = \sum_{i=0}^{n} a_ix^i,$$

where $a_n \neq 0$.
Instead of using the usual long division, Horner Division method (known also as Synthetic Division) is a faster computing strategy (shortcut) to use when

Key words and phrases: Polynomials, Division, Horner method, Generalized Horner method.
AMS (MOS) Subject Classifications: 13Pxx, 13P05.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

dividing the polynomial $P(x)$ by the monomial $x - c$, where $c \in \mathbb{R}$. The method is extremely handy when one manually needs to know the value of $P(c)$ for a wide range of values of c being a nesting technique that uses just n multiplications and n additions to evaluate an n^{th}-degree polynomial.

However, the history of Horner method is traced back to al-Nasaw (11^{th} century) who, for the first time, had used what would be later known as the Ruffini-Horner method (19^{th} century) to extract cube roots. A few years later, Al-Samaw'al al-Maghribi utilized the Ruffini-Horner approach to extract the fifth root of a sexagesimal positive integer number [2].

In our present age, Horner Division method has long been covered in college algebra courses as a required topic. However, this method of dividing a polynomial of degree n by a binomial of degree 1 is typically introduced in college algebra textbooks without mentioning whether it can be applied when the divisor is a polynomial of degree greater than 1. Some textbooks even go so far as to explicitly state that it is not applicable to such a divisor [3, 4].

It is a stroke of good fortune that I happened to be working on a paper presenting a generalization of the classic synthetic division which can be used to divide a polynomial by another polynomial of any degree [5].

Nonetheless, the aforementioned generalization is rather long and fragmented, considering the possible cases in which many operations are repeated. In fact, it is not a generalization of Horner's method, but rather a repeated imitation of that method.

In the present paper, we generalize Horner Division method to divide a polynomial by another polynomial of any degree.

2 Results

Let $A(x) = a_0 + a_1x + \cdots + a_nx^n$ and $B(x) = b_0 + b_1x + \cdots + b_mx^m$ be two polynomials of respective degrees n and m with $n \geq m$. In this section, we aim to generalize Horner Division by dividing A by B.

For convenience, we first assume that $b_m = 1$.

Fan [5] described the Horner algorithm generalization in terms of two different scenarios. However, it is not the precise generalization; rather, it is an iteration of Horner's method.

Our objective is to provide an algorithm that combines these two scenarios and provides a precise generalization of Horner's technique. To do so, we must first comprehend how the Euclidean division of polynomials works.

In the following, we summarize what happens in any long division problem:
If $A(x)$ and $B(x)$ are polynomials, with $B(x) \neq 0$ and $deg(A(x)) \geq deg(B(x))$, then there exist unique polynomials $Q(x)$ and $R(x)$ such that

$$A(x) = Q(x)B(x) + R(x),$$

where $degQ(x) = deg(A(x)) - deg(B(x))$, and $degR(x)deg(B(x))$.

The fundamental principle of the polynomial division algorithm is as follows:
If the leading coefficient of the divisor equals one (or is invertible), and the dividend has a degree greater than the divisor, we can scale the divisor to have the same degree and leading coefficient as the dividend, then subtract it from the dividend to eliminate the dividend's leading term. We then repeat this process for the remainder of the dividend, which has a smaller degree (because we eliminated the lead), and so on.

$$\overbrace{\left(a_n x^{k+m} + A_1(x)\right)}^{dividend} - a_n x^k \overbrace{\left(x^m + B_1(x)\right)}^{divisor}) = A(x) - a_n x^k B(x)$$

then

$$\frac{a_n x^{k+m} + A_1(x)}{x^m + B_1(x)} = a_n x^k + \underbrace{\frac{A(x) - a_n x^k B(x)}{x^m + B_1(x)}}_{recurse\ on\ this}$$

where the first equation is divided by $x^m + B_1(x)$ to produce the second equation. The long division procedure for polynomials is simply a tabular structure of the process produced by repeating this descending process until the dividend has a lower degree than the divisor.

Let us see how this division works when $B(x) = x + a$. As it is shown, we have to determine $n-1$ coefficients of the quotient $Q(x)$ and 1 coefficient of $R(x)$. We then have to achieve n operations. Horner organizes these n operations in a table of n columns.

$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ $-$ $q_{n-1}x^n + aq_{n-1}x^{n-1}$	$x + a$ $\underbrace{q_{n-1}x^{n-1} + \cdots + q_1x + q_0}_{\text{Let } q_{n-1} = a_n}$
$(a_{n-1} - aq_{n-1})x^{n-1} + \cdots + a_1x + a_0$ The second row-operation impose q_{n-2} to be $a_{n-1} - aq_{n-1}$	Each coefficient q_i corresponds to a row-operation, $0 \leq i \leq n-1$.
$\vdots$	
The k^{th} row-operation impose q_{n-k} to be $a_{n-k+1} - aq_{n-k+1}$	
$\vdots$	
The n^{th} row-operation impose q_0 to be $a_1 - aq_1$	
The $(n+1)^{th}$ row-operation impose $R(x)$ to be $a_0 - aq_0$	

For synthesizing this division method, Horner's algorithm defines the sequence $q_k,\ k = -1, \cdots, n = 1$ by the relations:

$$\left\{ \begin{array}{l} q_{n-1} = a_n \\ b_k = a_{k+1} - aq_{k+1}, \quad k = -1, \cdots, n-1. \end{array} \right\}$$

where $q_{n-1}, \cdots, q_0$ define the coefficients of the quotient $Q(x)$ and $q_{-1} = R(x)$. Using this synthesis, one can use only the numbers for calculation and avoid the usage of the variables and reduce the number of multiplication operations.

This synthesis can also be represented by the table below:

$$\begin{array}{c|cccccc} & a_n & a_{n-1} & \cdots & \cdots & a_1 & \cdots a_0 \cdots \\ -a & \downarrow & & & & & \\ & \multicolumn{5}{c}{\underbrace{q_{n-1} \quad q_{n-2} = a_{n-1} - aq_{n-1} \quad \cdots \quad q_1 \quad q_0}_{\textit{the quotient}}} & \underbrace{q_{-1} = a_0 - aq_0}_{\textit{the remainder}} \end{array}$$

Then, we can summarize the Horner method in the following steps:

Step 1: Write down the coefficients of the dividend $(P(x)$ where zero is used as a placeholder for any missing variable term or constant.

Step 2: As you divide by $x - a,\ x = a$ is a root. Note this root on the next line.

Step 3: On the third line start by copying down the first coefficient.

Now repeat the following steps:

Step 4: Multiply the last coefficient on the third line with the root and write the result in the second line.

Step 5: Add the numbers of lines one and two and write the sum on the third line.

We can combine the second and third lines by calculating multiplication and addition together.

The last number on the third line is the remainder. It is exactly $P(a)$. As an example, let us find the remainder and the quotient of the Euclidean division of $P(x) = 3x^3 + 2x^2 - 5x - 10$ by $x - 2$, using the algorithm of Horner.

We have
$$2 \left| \begin{array}{cccc} 3 & 2 & -5 & -10 \\ \downarrow & & & \\ 3 & 8 & 11 & 12 \end{array} \right.$$

So the remainder is 12 and the quotient is $3x^2 + 8x + 11$.

Similarly, let us now consider $B(x) = x^m + b_{m-1}mx^{m-1} \cdots + b_1x + b_0$ and see how the classic Euclidean division works. As it is known, we have to determine $n - m + 1$ coefficients of the quotient $Q(x)$ and m coefficient of $R(x)$. Then we have to achieve $n + 1$ operations.

$\cdots \quad a_m \quad a_{m-1} \; \cdots \; a_{n-m+1} \quad a_{n-m} \quad \cdots \quad a_1 \quad a_0$	$b_m \quad b_{m-1} \cdots b_{n-m} \cdots b_{2m-n} \cdots b_1 \quad b_0$
$q_1 b_{m-1} \quad q_0 b_{m-1}]$	$q_{n-m} \quad q_{n-m-1} \quad q_{n-m-2} \quad \cdots \quad q_0$
$q_2 b_{m-2} \quad q_1 b_{m-2} \cdots]$	
$\cdots \quad \cdots \quad \cdots \quad \cdots$	
$\cdots \quad \cdots \quad \cdots \quad \cdots$	
$\cdots \quad \cdots \quad \cdots \quad q_0 b_{n-m}]$	
$\cdots \quad \cdots \quad \cdots \quad \cdots$	
$-[q_{n-m} b_{2m-n} \quad \cdots \quad \cdots \quad \cdots \quad \cdots$	
$\cdots \quad \cdots \quad \cdots \quad \cdots$	
$-[q_{n-m} b_1 \quad q_{n-m-1} b_1 \quad \cdots \quad q_0 b_1]$	
$-[q_{n-m} b_0 \quad \cdots \quad q_1 b_0 \quad q_0 b_0$	
$r_{m-1} \; \cdots \; r_{n-m-1} \quad r_{n-m} \quad \cdots \quad r_1 \quad r_0$	

To synthesize this Euclidean division, we can use the same approach of Horner's algorithm. Then, let us define the sequences $q_{n-m-k},\ k = 0, \cdots, n-1$ and $r_l,\ l = 0, \cdots, m-1$ by the relations:

$$\left\{ \begin{array}{ll} q_{n-m-k} = a_{n-k} - \displaystyle\sum_{i=0}^{\inf\{m,k+1\}} q_{n-m-k+1+i} b_{m-i}, & k = 0, \cdots, n-1 \\ r_l = a_l - \displaystyle\sum_{i=0}^{\inf\{l+1,n-m\}} b_{l+1-i} q_i, & l = 0, \cdots, m-1, \end{array} \right\}$$

where $q_{n-m}, \cdots, q_0$ define the coefficients of the quotient $Q(x)$ and $r_{m-1}, \cdots, r_0$ define the coefficients of the remainder $R(x)$.

Under this synthesis, by avoiding the use of variables and limiting the amount of multiplication operations, one can use only the numbers for calculation.

This synthesis can also be represented through the following table:

$$\begin{array}{c|cccccc|c} & a_n & a_{n-1} & \cdots & \cdots & & a_m & a_{m-1} \cdots\cdots \cdots \quad \cdots\ a_0 \\ -b_{m-1} & \downarrow & & & & & & \\ -b_{m-2} & \multicolumn{6}{c|}{\underbrace{q_{n-m} \quad q_{n-2} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad q_1 \quad q_0}_{\textit{the quotient}}} & \underbrace{r_{m-1}, \cdots, \cdots\cdots\cdots r_0}_{\textit{the remainder}} \\ \cdots & & & & & & & \\ -b_1 & & & & & & & \\ -b_0 & & & & & & & \end{array}$$

Then, we can summarize the above generalization of Horner method in the following steps:

Step 1: This generalized Horner method consists in preparing a table of calculation: It is a table made up of two lines of which the number of columns equal to the degree of dividend +1.

Then, the first step is to place in the first line, from left to right, the coefficients of the dividend ($P(x)$ (written according to decreasing powers), where zero (0) is used as a placeholder for any missing variable term or constant.

Step 2: To the left of this main table, we will have a vertical column where we place the opposites of the divisor coefficients, which must be a monic polynomial, (written according to decreasing powers) from top to bottom, with the exception of the coefficient of the highest degree, where zero

(0) is used as a placeholder for any missing variable term or constant.

Step 3: We count the number of columns by the number of degree of the divisor from the right and we draw a vertical line called the separation line (this line is the separation wall between the coefficients of the quotient and the coefficients of the remainder).

Step 4: We go down with the first coefficient on the left.

Now repeat the following steps:

Step 5: Multiply, then, the first coefficients with the first terms of the vertical column, and add it to the following coefficient, then place the sum in the next box of the second line.

The calculation is completed without counting the coefficients products obtained on the right of the separation line.

3 Applications

1. As an example, let's find the remainder and the quotient of the Euclidean division of $A = 2x^8 + x^5 - 3x^2 + 4x - 1$ by $B = x^4 + 2x - 1$.

   In this case, it is equal to: $8 + 1 = 9$.

   Step 1:

$$\begin{array}{c|ccccccccc} & 2 & 0 & 0 & 1 & 0 & 0 & -3 & 4 & -1 \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \end{array}$$

   Step 2:

$$\begin{array}{r|ccccccccc} & 2 & 0 & 0 & 1 & 0 & 0 & -3 & 4 & -1 \\ 0 & & & & & & & & & \\ 0 & & & & & & & & & \\ -\ \ 2 & & & & & & & & & \\ 1 & & & & & & & & & \end{array}$$

   Step 3:

	2	0	0	1	0	0	−3	4	−1
0									
0									
− 2									
1									

Step 4:

	2	0	0	1	0	0	−3	4	−1
0	2								
0									
− 2									
1									

Step 5:

	2	0	0	1	0	0	−3	4	−1
0	2	0	0	−3	2	0	3	−3	1
0									
− 2									
1									

Thus, the Euclidean division is complete.

We obtain:

$$Q(x) = 2x^4 - 3x + 2,\ \ R(x) = 3x^2 - 3x + 1$$

and so:

$$A(x) = (2x^4 - 3x + 2)B(x) + 3x^2 - 3x + 1.$$

2. For another example, let us find the remainder and the quotient of the Euclidean division of $A(x) = 2x^8+x^5-3x^2+4x-1$ by $B(x) = x^5+2x-1$.

   Then

	2	0	0	1	0	0	−3	4	−1
0	2	0	0	1	−4	2	−3	2	0
0									
0									
− 2									
1									

and then, the Euclidean division of $A(x)$ by $B(x)$ is :

$$A(x) = (2x^3 + 1)B(x) - 4x^4 + 2x^3 - 3x^2 + 2x.$$

3. let us find the remainder and the quotient of the Euclidean division of $A = 2x^8 + x^5 - 3x^2 + 4x - 1$ by $B = x^3 + 2x - 1$.

   Then we have

		2	0	0	1	0	0	−3	4	−1
	0	2	0	−4	3	8	−10	−16	32	−11
−	2									
	1									

   and so:

$$A(x) = (2x^5 - 4x^3 + 3x^2 + 8x - 11)B(x) - 16x^2 + 32x - 11.$$

4 Conclusion

We are accustomed to using the long division approach while performing a Euclidean division on a polynomial with a divisor of degree ≥ 2. That is because it was widely held that Horner's approach could only be applied if the divisor was of the type $x - a$ and could not be generalized for higher degree.

We have generalized the Horner method to allow you division of any arbitrary polynomial, using the Horner approach, not only monomials, giving you a more general and valuable skill than long division, which can helpful in comprehending CRCs and Galois fields.

Since our generalization reduces the number of multiplication operations, lengthy division no longer has any practical advantages. You can see that by structuring your program to execute Euclidean division of two polynomials using our generalization, the computer execution time will be much improved. The processing time for a single multiplication step is $5to20$ times that of an addition procedure.

One benefit of using our generalization in practice is that polynomial division does not require the entire writing down of all intermediate terms. Additionally, if that component is causing you errors, negating the divisor so you add instead of subtracting can be useful. So it is conceivable to use a hybrid strategy that benefits from some of the advantages of this generalization.

5 Acknowledgments

The authors wish to thank the referees for their thoughtful comments. The authors would like to thank the Deanship of Scientific Research at Umm Al-Qura University for supporting this work by Grant Code: (22UQU 4350388DSR02).

References

[1] William George Horner, A new method of solving numerical equations of all orders, by continuous approximation, Philosophical Transactions, Royal Society of London, **109,** (1819), 308–335.

[2] A. Abbassi, Root extraction by Nizām al-Dīn al-Nīsābūrī (d. ca. 1330/730AH), Kuwait Journal of Science, **49,** no. 2, (2022).

[3] R. Larson, R. P. Hostetler, B. H. Edwards, D. C. Falvo, Precalculus functions and graphs: A graphing approach, Houghton Mifflin Company, (1997).

[4] M. L. Lial, E. J. Hornsby, Algebra for college students, Addison Wesley Publishing Company, (1999).

[5] L. Fan, A generalization of synthetic division and a general theorem of division of polynomials, Mathematical Medley, **30,** no. 1, (2003), 30–37.