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On the Diophantine equation $147^x + 741^y = z^2$

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Abstract

In this paper, we prove that the Diophantine equation $147^x + 741^y = z^2$ has no non-negative integer solution.

1 Introduction

In 2013, Sroysang [1] showed that the Diophantine equation $23^x + 32^y = z^2$ has no non-negative integer solution. In 2014, Sroysang [2] proved that the Diophantine equation $46^x + 64^y = z^2$ has no non-negative integer solution. In 2016, Srisarakham and Thongmoon [3] showed that $(1, 0, 7)$ is a unique non-negative integer solution for the Diophantine equation $48^x + 84^y = z^2$. In 2019, Sugandha et al. [4] verified that the Diophantine equation $13^x + 31^y = z^2$ has no non-negative integer solution.

In this paper, we study the Diophantine equation $147^x + 741^y = z^2$ where x, y and z are non-negative integers.

2 Preliminaries

Throughout this paper, $a \equiv_m b$ always means a is congruent to b modulo m where a, b, m are integers such that $m \geqslant 1$. Moreover, we will write $a \equiv_m b, c$

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to mean that $a \equiv_m b$ or $a \equiv_m c$.

Now, we shall recall the Catalan's conjecture [5] of 1844 which was proved by Mihailescu [6] in 2004.

Theorem 2.1 (Catalan's conjecture). *The Diophantine equation $a^x - b^y = 1$ has the unique solution $(a, b, x, y) = (3, 2, 2, 3)$ where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$.*

Next, we need the following lemmas.

Lemma 2.2. *The Diophantine equation $147^x + 1 = z^2$ has no non-negative integer solution where x, y, z are non-negative integers.*

Proof. Assume that there exist non-negative integers x and z such that $147^x + 1 = z^2$. If $x = 0$, $z^2 = 2$, which is a contradiction. Now, we have $x \geqslant 1$. By Theorem 2.1, $x = 1$. Then $z^2 = 148$ which is impossible. This finishes the proof. □

Lemma 2.3. *The Diophantine equation $1 + 741^y = z^2$ has no non-negative integer solution where x, y, z are non-negative integers.*

Proof. The proof is similar to the proof of Lemma 2.2. □

Lemma 2.4. [7] *If z is an integer, then $z^2 \equiv_{19} 0, 1, 4, 5, 6, 7, 9, 11, 16, 17$.*

3 Main Results

Now, we shall prove a lemma which will be useful in the main theorem.

Lemma 3.1. *If x is a positive odd integer, then $14^x \equiv_{19} 2, 3, 8, 10, 12, 13, 14, 15, 18$.*

Proof. We will prove by induction that $14^{2n-1} \equiv_{19} 2, 3, 8, 10, 12, 13, 14, 15, 18$ for all $n \in \mathbb{N}$. If $n = 1$, we have $14^1 \equiv_{19} 14$. Thus the statement is true for $n = 1$. Assume that it is true for $n = k$. Then $14^{2k-1} \equiv_{19} 2, 3, 8, 10, 12, 13, 14, 15, 18$, and so $14^{2k+1} \equiv_{19} 12, 18, 10, 3, 15, 2, 8, 14, 13$. Thus the statement is true for $n = k + 1$ as desired. □

Next, we shall give the main result.

Theorem 3.2. *The Diophantine equation $147^x + 741^y = z^2$ has no non-negative integer solution where x, y, z are non-negative integers.*

Proof. Assume that there exist non-negative integers x, y, z such that $147^x + 741^y = z^2$. By Lemmas 2.2 and 2.3, $x \geqslant 1$ and $y \geqslant 1$. If x is even, then $z^2 = 147^x + 741^y \equiv_4 2$, which contradicts with the fact that $z^2 \equiv_4 0, 1$. Then x is odd. Since $147 \equiv_{19} 14$, by Lemma 3.1, we have $147^x \equiv_{19} 2, 3, 8, 10, 12, 13, 14, 15, 18$. Since $741^y \equiv_{19} 0$, we obtain that $z^2 \equiv_{19} 2, 3, 8, 10, 12, 13, 14, 15, 18$, which contradicts with Lemma 2.4. This completes the proof. □

4 Conclusion

In this paper, we showed that the Diophantine equation $147^x + 741^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers.

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