Eigensharp Property of Some Certain Graphs and their Complements

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Abstract

The biclique partition number of a graph $G$, denoted by $bp(G)$, is the minimum number of complete bipartite subgraphs are needed to be representing all edges of $G$. A graph $G$ is called an eigensharp graph if it is satisfying $bp(G) = \max\{i_-(A(G)), i_+(A(G))\}$ where $i_-(A(G))$ and $i_+(A(G))$ are the number of negative and positive eigenvalues of the adjacency matrix of $G$, respectively. In this paper, we are interested in studying some special graphs in terms of having the property of eigensharp. We show that the barbell graph, the sun graph, the friendship graph and their complements have the property of eigensharp.

Key words and phrases: Complete bipartite graph, Biclique, Biclique partition number, Barbell graph, Sun graph, Friendship graph.

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1 Introduction

Graph theory as a branch of mathematics is simply a network of points connected by lines. In this regard, graph theory comes up frequently in applied science. For instance, graph theory plays an important role in robotic navigation [2], the optimization of threat-detecting sensors [3], and chemical data classification [4]. Recently, the study of graph theory has attracted the attention of researchers studying new concepts. One of the interesting concepts of graph theory is graph covering, particularly the biclique partition number. One motivation for looking at this parameter is to minimize storage space. Listing no more subgraphs of the minimal complete 2-part decomposition of $G$ are needed arrange as an adjacency list representation.

Covering a graph with subgraphs from a special graph family is an important concept. Much work has been done on several types of graph covering including path covering, tree covering, cycle covering and star covering [2], [3], and [6].

A biclique is a complete bipartite subgraph. A biclique covering of a graph is a family of biclique subgraphs, say $R_1, R_2, \ldots, R_n$ having the property that each edge of $G$ is containing in at least one biclique subgraph $R_i$ for some $i$. A biclique partition of a graph $G$ is a collection of bicliques of $G$, which together contain each edge of $G$ exactly once. The smallest cardinality of any biclique partition of a graph $G$ is called the biclique partition number of $G$ and is denoted by $bp(G)$.

The biclique partition number has numerous applications in various fields of applied science, such as computational complexity, automata and language theories, partial orders, artificial intelligence, and geometry see, for example, [5].

The biclique partition number was brought up for the first time by Graham and Pollak [3] in 1971. They were motivated by a network addressing problem. For more details about graph addressing, please see [3] and [5].

In this paper all graphs are finite, undirected, and simple. For a graph $G = (V(G), E(G))$, $V(G)$ is the vertex set and $E(G)$ is the edge set. The order of a graph $G$ is equal to the cardinality of $V(G)$ and is denoted by $|G|$. The complement of $G$, denoted by $\overline{G}$, satisfies $u$ is adjacent to $v$ in $G$ if and only if $u$ and $v$ are non-adjacent in $\overline{G}$.

Assume $A(G)$ to be the adjacency matrix of a graph $G$. If $\lambda_i : 1 \leq i \leq k$, are the distinct eigenvalues of $A(G)$ with multiplicity $m_i$, then $\sigma(G) = \left( \begin{array}{cccc} \lambda_1 & \lambda_2 & \ldots & \lambda_k \\ m_1 & m_2 & \ldots & m_k \end{array} \right)$ is called the spectrum of $G$. The eigenvalues of the
adjacency matrix $A(G)$ are all real, since $A(G)$ is a symmetric matrix. For example, the spectrum of complete graph $K_n$ is
\[
\sigma(K_n) = \left( \begin{array}{cc} n-1 & -1 \\ 1 & n-1 \end{array} \right)
\]
and the spectrum of complete bipartite graph $K_{n,m}$ is
\[
\sigma(K_{n,m}) = \left( \begin{array}{cc} \sqrt{nm} & 0 & -\sqrt{nm} \\ 1 & nm-2 & 1 \end{array} \right).
\]

For a graph $G$ the number of positive, negative and zero eigenvalues are denoted by $i_+(A(G))$, $i_-(A(G))$ and $i_0(A(G))$ respectively. It can be proven that $i_+(A(G)) > 0$ and $i_-(A(G)) > 0$ for any non-null graph $G$.

Witsenhausen [5] showed that in the case of a graph $G$ the biclique partition number $bp(G) \geq \max\{i_+(A(G)), i_-(A(G))\}$. A graph $G$ is called an eigensharp when $bp(G) = \max\{i_+(A(G)), i_-(A(G))\}$.

There are certain families of eigensharp graphs such as complete graphs $K_n$, complete bipartite graphs $K_{n,m}$, and cycle graphs $C_n$ with $n \neq 4k$, $k \geq 2$. In fact, $bp(K_{n,m}) = 1$, $bp(K_n) = n - 1$, and $bp(C_n) = \left\lceil \frac{n}{2} \right\rceil$ (see [4] and [5]).

The following two lemmas are useful to prove our main results.

**Lemma 1.1.** [8]. Let $B$, $C$ and $D$ be matrices and assume that $B$ is nonsingular. Then
\[
i_\pm \left[ \begin{array}{cc} B & C \\ C^* & D \end{array} \right] = i_\pm(B) + i_\pm(D - C^*B^{-1}C),
\]
where $C^*$ is the conjugate matrix of the matrix $C$.

**Lemma 1.2.** [1] Let $G$ be a graph obtained by joining a vertex $a$ in a graph $G_1$ and a vertex $b$ in a graph $G_2$ by an edge $e$. Let $G_1 - a, G_2 - b$ be the induced subgraphs of $G_1$ and $G_2$ after deleting the vertices $a$ and $b$ respectively. Then, the characteristic polynomial $P_G(\lambda) = P_{G_1}(\lambda)P_{G_2}(\lambda) - P_{G_1-a}(\lambda)P_{G_2-b}(\lambda)$.

In this paper, we present the biclique partition number in several graph classes that have never been examined. We study the biclique partition number of the barbell graph $B_n$ and its complement, the Sun graph $S(n)$ and its complement and the friendship $F_n$ graph and its complement. Also we show that these graphs have the property of eigensharp.

### 2 The Barbell graph and its Complement.

In this section, we determine the biclique partition number of the barbell graph $B_n$ and its complement $\overline{B_n}$. Moreover, we show that the graphs $B_n$
and \( B_n \) are eigensharp for each \( n \geq 2 \). The \( n \)-barbell graph \( B_n \) is a graph with \( 2n \) vertices designed by connecting two copies of a complete graph of degree \( n \) by a bridge. Suppose that \( \{u_i : 1 \leq i \leq n\} \) is the vertex set of the complete graph \( K^1_n \) and \( \{v_j : 1 \leq j \leq n\} \) is the vertex set of \( K^2_n \). Assume \( e \) is the edge connecting \( K^1_n \) and \( K^2_n \). Then, the vertex set of \( B_n \) is given by \( V(B_n) = \{u_i, v_j : 1 \leq i, j \leq n\} \) and the edge set of \( B_n \) is \( E(B_n) = \{e, u_r u_s, v_k v_t : 1 \leq r < s \leq n, 1 \leq k < t \leq n\} \).

**Theorem 2.1.** The barbell graph, \( B_n \), is an eigensharp graph for each \( n \geq 2 \).

**Proof.** If \( n = 2 \), then \( B_2 \simeq P_4 \), and hence \( \sigma(B_2) = \left(\frac{1 + \sqrt{5}}{2}, \frac{-1 + \sqrt{5}}{2}\right) \). Thus, \( bp(B_2) = 2 = \max\{i_+(A(B_2)), i_-(A(B_2))\} \).

For \( n \geq 3 \), let \( G_1 = K^1_n \), \( G_2 = K^2_n \) and let \( e = u_i v_j \) be the bridge connecting \( G_1 \) and \( G_2 \). Then \( G_1 - u_i \) and \( G_2 - v_j \) are isomorphic to the complete graph \( K_{n-1} \) and so, by Lemma 1.2, we have

\[
P_\lambda(B_n) = (P_\lambda(K_n))^2 - (P_\lambda(K_{n-1}))^2.
\]

Therefore,

\[
P_\lambda(B_n) = (\lambda + 1)^{2n-2} - (\lambda - n + 1)^2 - (\lambda + 1)^{2n-4}(\lambda - n + 2)^2 = (\lambda + 1)^{2n-4}(\lambda^2 + (3 - n)\lambda - 2n + 3)(\lambda^2 + (1 - n)\lambda - 1).
\]

Hence the spectrum of the barbell graph \( B_n \) is

\[
\sigma(B_n) = \left(\frac{-1}{2n-4}, \frac{n^2 + 2n - 3}{1}, \frac{n^2 - 2n + 5}{1}\right)
\]

with \( \left(\frac{n^2 + 2n - 3}{1}\right) \) and \( \left(\frac{n^2 - 2n + 5}{1}\right) \) are positive values, and \( \left(\frac{n^2 - 2n + 3}{1}\right) \) and \( \left(\frac{n^2 - 2n + 5}{1}\right) \) are negative values for \( n \geq 3 \). Therefore \( \max\{i_+(A(B_n)), i_-(A(B_n))\} = 2n - 2 \) and hence, \( bp(B_n) \geq 2(n - 1) \).

Now, to show that \( bp(B_n) = 2(n - 1) \), let \( V(K^1_n) = \{u_i : 1 \leq i \leq n\} \) and \( V(K^2_n) = \{v_j : 1 \leq j \leq n\} \). Assume that \( e = u_n v_1 \) is the bridge from \( K^1_n \) to \( K^2_n \). By Graham and Pollak Theorem \( bp(K^i_n) = n - 1 \) for each \( i = 1, 2 \). The edges of \( K^1_n \) are covered by the disjoint stars generated by \( \{u_2, u_3, \ldots, u_n\} \). The edges \( K^2_n \) are covered by the disjoint stars generated by \( \{v_2, v_3, \ldots, v_n\} \). Also the bridge \( e \) is covered by the star generated by the vertex \( u_n \). So, \( bp(B_n) \leq 2(n - 1) \) and therefore, \( B_n \) is an eigensharp graph with \( bp(B_n) = \max\{i_+(A(B_n)), i_-(A(B_n))\} = 2n - 2 \). \( \square \)
Theorem 2.2. The complement of the barbell graph is eigensharp.

Proof. Assume that \( V(K^1_n) = \{u_i : 1 \leq i \leq n\} \) and \( V(K^2_n) = \{v_j : 1 \leq j \leq n\} \) are the sets whose union constitutes the set of vertex of the graph \( B_n \). Suppose that \( e = u_iv_1 \) is the bridge connecting \( K^1_n \) and \( K^2_n \). Then, for the complement graph \( \overline{B}_n \), for each \( 1 < i \leq n \) and \( 1 < j \leq n \), we have \( u_iv_j \in E(\overline{B}_n) \) except \( e = u_iv_1 \notin E(\overline{B}_n) \).

Since the subgraphs induced by \( V(K^1_n) \) and \( V(K^2_n) \) are null graphs in \( \overline{B}_n \), we have \( \overline{B}_n \simeq K_{n,n} - e \). Hence the adjacency matrix of \( \overline{B}_n \) is \( A = \begin{bmatrix} 0 & Q \\ Q & 0 \end{bmatrix} \), where \( Q \) is the \( n \times n \) matrix with all entries are 1 except the entry \( q_{1,1} \) is zero.

Now, let \( B = \left( \begin{array}{cccc} 1 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 2 & \ldots & 2 \\ \end{array} \right)_{2 \times n} \), \( C = \begin{bmatrix} -1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{bmatrix}_{2 \times n} \) and let \( D = CB = \begin{bmatrix} n-2 & 2n-3 \\ 1 & 1 \end{bmatrix} \). Then \( Q = BC \) with spanning rank 2 and so, by [9], the characteristic polynomial of \( Q \) is \( P_Q(x) = x^{n-2}P_D(x) \). Hence,

\[
P_Q(x) = x^{n-2}(x^2 - (n-1)x - (n-1)),
\]

and so, \( \sigma(Q) = \left( \begin{array}{c} 0 \\ n-2 \end{array} \right) = \left( \begin{array}{c} (n-1) \pm \sqrt{n^2 + 2n - 3} \\ 1 \end{array} \right) \).

Now, \( \det(A - \lambda I) = \det(\lambda^2 I - Q^2) \); i.e., \( P_A(x) = P_Q(x)P_{-Q}(x) \) [7]. Thus, we have

\[
\sigma(\overline{B}_n) = \left( \begin{array}{cc} 0 & \pm \frac{1}{2} \left( (n-1) \pm \sqrt{n^2 + 2n - 3} \right) \\ 2n-4 & 1 \end{array} \right)
\]

By simple calculations, we conclude that \( n-1 < \sqrt{n^2 + 2n - 3} = \sqrt{(n-1)(n+3)} \); i.e., \( (n-1) - \sqrt{n^2 + 2n - 3} < 0 \) for \( n \geq 2 \).

Therefore, \( \max \{i_+(A(\overline{B}_n)), i_-(A(\overline{B}_n))\} = 2 \) and so \( bp(\overline{B}_n) \geq 2 \).

On the other hand, since \( \overline{B}_n \simeq K_{n,n} - e \), \( \overline{B}_n \) have a biclique partition consists of the subgraph \( K_{n-1,n} \) with vertex set \( V = \{u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\} \) and the star \( S \) generated by the vertex \( u_1 \). Therefore, \( bp(\overline{B}_n) = 2 \).

\[ \square \]

3 The Sun Graph and its Complement.

In this section, we study the biclique partition number of the sun graph \( S(n) \) and its complement. Moreover, we show that \( S(n) \) and \( S(n) \) are eigensharp
graphs for all \( n \geq 2 \).
The sun graph \( S(n) \) is a graph consisting of \( 2n \) vertices obtained by attaching \( n \) pendant edges to a complete graph \( K_n \). Let \( V(S(n)) = \{u_i, v_i : 1 \leq i \leq n\} \) be the vertex set of \( S(n) \) where, \( u_iu_j \in E(S(n)) \) for each \( 1 \leq i, j \leq n \) and \( u_iv_i \) is a pendent edge in \( S(n) \).

**Theorem 3.1.** The Sun Graph is eigensharp.

**Proof.** By the definition of \( E(S(n)) \), the adjacency matrix \( A(S(n)) = \begin{bmatrix} A(K_n) & I_n \\ I_n & 0 \end{bmatrix} \), where \( A(K_n) \) is the adjacency matrix of \( K_n \) and \( I_n \) is the identity matrix of size \( n \times n \). So, by Lemma 1 \( i_\pm \begin{bmatrix} A(K_n) & I_n \\ I_n & 0 \end{bmatrix} = i_\pm(A(K_n)) + i_\mp(-A(K_n^{-1})) = n \) which leads to \( bp(S(n)) \geq \max\{n, n\} = n \).

On the other hand, let \( W = \{S_1, ..., S_n\} \) be the set of \( n \) disjoint stars in \( S(n) \) generated by \( u_1, ..., u_n \) respectively. Then \( S_i \) contains at least the edge \( u_iv_i \) for each \( 1 \leq i \leq n \). Thus, \( W \) forms a biclique partition of \( S(n) \) with minimum cardinality equals \( n \). Hence, \( bp(S(n)) = n \) and so \( S(n) \) is an eigensharp graph.

Note that, the corona product \( K_n \odot K_1 \) has been shown that it has the eigensharp property [6]. Since the sungraph \( S(n) \) is isomorphic to \( K_n \odot K_1 \), this confirms that \( S(n) \) is eigensharp.

**Theorem 3.2.** The complement of the Sun Graph is eigensharp.

**Proof.** Since the adjacency matrix of \( S(n) \) is \( A(S(n)) = \begin{bmatrix} A(K_n) & A(K_n) \\ A(K_n) & 0 \end{bmatrix} \), then by Lemma 1, \( i_\pm \begin{bmatrix} A(K_n) & A(K_n) \\ A(K_n) & 0 \end{bmatrix} = i_\pm(A(K_n)) + i_\mp(A(K_n)) \). Thus \( \max\{i_+(A(S(n))), i_-(A(S(n)))\} = \max\{n, n\} = n \).

Now, let \( H = \{S_1, S_2, ..., S_n\} \) be the set of \( n \) disjoint stars in \( S(n) \) generated by \( v_1, ..., v_n \) respectively. Since the graph induced by \( \{u_1, ..., u_n\} \) is a null subgraph, \( H \) forms a biclique partition of \( S(n) \) with minimum cardinality equals \( n \). Thus, \( bp(S(n)) = n \) and so \( S(n) \) is an eigensharp graph.
4 The Friendship Graph and its Complement

In this section, we study the biclique partition number of the friendship graph $F_n$ and its complement $F_n^c$. Moreover, we show that the graphs $F_n$ and $F_n$, are eigensharp graphs for all $n \geq 2$.

The friendship graph $F_n$ is the joint of a one vertex $v$ with disjoint $n$ path graphs $K_2$. Let $V(F_n) = \{v, u_i : 1 \leq i \leq 2n\}$ be the vertex set of $F_n$. Then $E(F_n) = \{vu_i, 1 \leq i \leq 2n\} \cup \{u_{2j-1}u_{2j}, 1 \leq j \leq n\}$.

**Theorem 4.1.** The Friendship graph is eigensharp for $n \geq 2$.

*Proof.* Let $H_1$ be the subgraph of $F_n$ with a one vertex $v$; let $H_2$ be the graph induced by the disjoint union of the $n$ path graphs $K_2$. Then $F_n = H_1 \cup H_2$ with $i_-(A(H_1)) = i_+(A(H_1)) = 0$ and

$$A(H_2) = \begin{bmatrix}
A(K_2) & 0 & \cdots & 0 \\
0 & A(K_2) & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A(K_2)
\end{bmatrix}_{2n \times 2n}$$

thus, $i_-(A(H_2)) = i_+(A(H_2)) = n$. Since each of $H_1$ and $H_2$ are regular graphs, by [2], we have $i_-(A(F_n)) = n + 1$ and $i_+(A(F_n)) = n - 1$ and hence $bp(F_n) \geq n + 1$. Now by the definition of $E(F_n)$, one can see that the set $B = \{vu_i, 1 \leq i \leq 2n\}$ is a star and the set $D^j = \{u_{2j-1}u_{2j}, 1 \leq j \leq n\}$ is a complete bipartite graph $K_{1,1}$, thus the set $\{B, D^1, D^2, \ldots, D^n\}$ is a biclique partition of $F_n$ with minimum cardinality $n + 1$. Hence, $bp(F_n) = n + 1$, so $F_n$ is an eigensharp graph. \hfill \Box

**Theorem 4.2.** The complement of friendship graph is eigensharp for $n \geq 2$.

*Proof.* Let $v$ be the vertex in $F_n$ join with disjoint $n$ path graphs $K_2$ induced by the set of vertex $\{u_i : 1 \leq i \leq 2n\}$. Thus, $F_n^c$, the complement of $F_n$, is decomposed by two components $K_1$ and $W$, where $K_1 = v$ and $W$ is a graph of order $2n$ in which any pair of distinct vertices $u_iu_j$ forms an edge with the exception of the pair $\{u_{2j-1}u_{2j}, 1 \leq j \leq n\}$. Therefore, $\sigma(F_n^c) = \{0\} \cup \sigma(W)$, where the adjacency matrix of $W$ is $A(W) = \begin{bmatrix} A(K_n) & A(K_n) \\
A(K_n) & A(K_n) \end{bmatrix}$. Thus, by
Lemma 1, $i_\pm(A(W)) = i_\pm(A(K_n))$; then

$$bp(F_n) \geq \max\{i_+(A(K_n)), i_-(A(K_n))\} = n - 1.$$ 

On the other hand, suppose that $X_1 = \{u_1, u_2\}, Y_1 = \{u_3, u_4, \ldots, u_{2n}\}$, $X_2 = \{u_3, u_4\}, Y_2 = \{u_5, u_6, \ldots, u_{2n}\}, \ldots, X_{n-1} = \{u_{2n-3}, u_{2n-2}\}, Y_{n-1} = \{u_{2n-1}, u_{2n}\}$. Let $K(X_s, Y_s)$ be the complete bipartite subgraph with two bipartite sets $X_s$ and $Y_s, 1 \leq s \leq n-1$. Then $H = \{K(X_s, Y_s) : 1 \leq s \leq n - 1\}$ is a set of biclique partition of a graph $F_n$ such that $K(X_s, Y_s) \cong K_{2,2n-2s}$. Since $H$ is a biclique partition of $F_n$ with cardinality $n - 1$, $bp(F_n) = n - 1$ and so $F_n$ is an eigensharp graph.

References


