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Eigensharp Property of Some Certain Graphs and their Complements

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Abstract

The biclique partition number of a graph G, denoted by $bp(G)$, is the minimum number of complete bipartite subgraphs are needed to be representing all edges of G. A graph G is called an eigensharp graph if it is satisfying $bp(G) = \max\{i_-(A(G)), i_+(A(G))\}$ where $i_-(A(G))$ and $i_+(A(G))$ are the number of negative and positive eigenvalues of the adjacency matrix of G, respectively. In this paper, we are interested in studying some special graphs in terms of having the property of eigensharp. We show that the barbell graph, the sun graph, the friendship graph and their complements have the property of eigensharp.

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1 Introduction

Graph theory as a branch of mathematics is simply a network of points connected by lines. In this regard, graph theory comes up frequently in applied science. For instance, graph theory plays an important role in robotic navigation [2], the optimization of threat-detecting sensors [3], and chemical data classification [4]. Recently, the study of graph theory has attracted the attention of researchers studying new concepts. One of the interesting concepts of graph theory is graph covering, particularly the biclique partition number. One motivation for looking at this parameter is to minimize storage space. Listing no more subgraphs of the minimal complete 2-part decomposition of G are needed arrange as an adjacency list representation.

Covering a graph with subgraphs from a special graph family is an important concept. Much work has been done on several types of graph covering including path covering, tree covering, cycle covering and star covering [2], [3], and [6].

A biclique is a complete bipartite subgraph. A biclique covering of a graph is a family of biclique subgraphs, say $R_1, R_2, \ldots, R_n$ having the property that each edge of G is containing in at least one biclique subgraph R_i for some i. A biclique partition of a graph G is a collection of bicliques of G, which together contain each edge of G exactly once. The smallest cardinality of any biclique partition of a graph G is called the biclique partition number of G and is denoted by $bp(G)$.

The biclique partition number has numerous applications in various fields of applied science, such as computational complexity, automata and language theories, partial orders, artificial intelligence, and geometry see, for example, [5].

The biclique partition number was brought up for the first time by Graham and Pollak [3] in 1971. They were motivated by a network addressing problem. For more details about graph addressing, please see [3] and [5].

In this paper all graphs are finite, undirected, and simple. For a graph $G = (V(G),\ E(G))\,, V\ (G)$ is the vertex set and $E(G)$ is the edge set. The order of a graph G is equal to the cardinality of $V(G)$ and is denoted by $|G|$. The complement of G, denoted by $\overline{G}$, satisfies u is adjacent to v in G if and only if u and v are non-adjacent in $\overline{G}$.

Assume $A(G)$ to be the adjacency matrix of a graph G. If $\lambda_i : 1 \leq i \leq k$, are the distinct eigenvalues of $A(G)$ with multiplicity m_i, then $\sigma(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & ... & \lambda_k \\ m_1 & m_2 & ... & m_k \end{pmatrix}$ is called the spectrum of G. The eigenvalues of the adjacency matrix $A(G)$ are all real, since $A(G)$ is a symmetric matrix. For example, the spectrum of complete graph K_n is $\sigma(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$ and the spectrum of complete bipartite graph $K_{n,m}$ is $\sigma(K_{n,m}) = \begin{pmatrix} \sqrt{nm} & 0 & -\sqrt{nm} \\ 1 & nm-2 & 1 \end{pmatrix}$.

For a graph G the number of positive, negative and zero eigenvalues are denoted by $i_+(A(G))$, $i_-(A(G))$ and $i_0(A(G))$ respectively. It can be proven that $i_+(A(G)) > 0$ and $i_-(A(G)) > 0$ for any non-null graph G.

Witsenhausen [5] showed that in the case of a graph G the biclique partition number $bp(G) \geq \max\{i_+(A(G)),\ i_-(A(G))\}$. A graph G is called an eigensharp when $bp(G) = \max\{i_+(A(G)),\ i_-(A(G))\}$.

There are certain families of eigensharp graphs such as complete graphs K_n, complete bipartite graphs $K_{n,m}$, and cycle graphs C_n with $n \neq 4k$, $k \geq 2$. In fact, $bp(K_{n,m}) = 1$, $bp(K_n) = n-1$, and $bp(C_n) = \lceil \frac{n}{2} \rceil$ (see [4] and [5]).

The following two lemmas are useful to prove our main results.

Lemma 1.1. *[8]. Let B, C and D be matrices and assume that B is non-singular. Then*

$$i_\pm \begin{bmatrix} B & C \\ C^* & D \end{bmatrix} = i_\pm(B) + i_\pm(D - C^* B^{-1} C),$$

where C^* is the conjugate matrix of the matrix C.

Lemma 1.2. *[1] Let G be a graph obtained by joining a vertex a in a graph G_1 and a vertex b in a graph G_2 by an edge e. Let $G_1 - a, G_2 - b$ be the induced subgraphs of G_1 and G_2 after deleting the vertices a and b respectively. Then, the characteristic polynomial $P_G(\lambda) = P_{G_1}(\lambda)P_{G_2}(\lambda) - P_{G_1-a}(\lambda)P_{G_2-b}(\lambda)$.*

In this paper, we present the biclique partition number in several graph classes that have never been examined. We study the biclique partition number of the barbell graph B_n and its complement, the Sun graph $S(n)$ and its complement and the friendship F_n graph and its complement. Also we show that these graphs have the property of eigensharp.

2 The Barbell graph and its Complement.

In this section, we determine the biclique partition number of the barbell graph B_n and its complement $\overline{B_n}$. Moreover, we show that the graphs B_n

and $\overline{B_n}$ are eigensharp for each $n \geq 2$. The $n-$barbell graph B_n is a graph with $2n$ vertices designed by connecting two copies of a complete graph of degree n by a bridge. Suppose that $\{u_i : 1 \leq i \leq n\}$ is the vertex set of the complete graph K_n^1 and $\{v_j : 1 \leq j \leq n\}$ is the vertex set of K_n^2. Assume e is the bridge connecting K_n^1 and K_n^2. Then, the vertex set of B_n is given by $V(B_n) = \{u_i, v_j : 1 \leq i, j \leq n\}$ and the edge set of B_n is $E(B_n) = \{e, u_r u_s, v_k v_t : 1 \leq r < s \leq n, 1 \leq k < t \leq n\}$.

Theorem 2.1. *The barbell graph, B_n is an eigensharp graph for each $n \geq 2$.*

Proof. If $n = 2$, then $B_2 \simeq P_4$, and hence $\sigma(B_2) = \begin{pmatrix} \frac{1\pm\sqrt{5}}{2} & \frac{-1\pm\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}$. Thus, $bp(B_2) = 2 = \max\{i_+(A(B_2)), i_-(A(B_2))\}$.

For $n \geq 3$, let $G_1 = K_n^1, G_2 = K_n^2$ and let $e = u_i v_j$ be the bridge connecting G_1 and G_2. Then $G_1 - u_i$ and $G_2 - v_j$ are isomorphic to the complete graph K_{n-1} and so, by Lemma 1.2, we have

$$P_\lambda(B_n) = (P_\lambda(K_n))^2 - (P_\lambda(K_{n-1}))^2.$$

Therefore,

$$\begin{aligned} P_\lambda(B_n) &= (\lambda+1)^{2n-2}(\lambda-n+1)^2 - (\lambda+1)^{2n-4}(\lambda-n+2))^2 \\ &= (\lambda+1)^{2n-4}(\lambda^2+(3-n)\lambda-2n+3)(\lambda^2+(1-n)\lambda-1). \end{aligned}$$

Hence the spectrum of the barbell graph B_n is

$$\sigma(B_n) = \begin{pmatrix} -1 & \frac{n-3}{2} \pm \frac{1}{2}\sqrt{n^2+2n-3} & \frac{n-1}{2} \pm \frac{1}{2}\sqrt{n^2-2n+5} \\ 2n-4 & 1 & 1 \end{pmatrix}$$

with $\left(\frac{n-3}{2} + \frac{1}{2}\sqrt{n^2+2n-3}\right)$ and $\left(\frac{n-1}{2} + \frac{1}{2}\sqrt{n^2-2n+5}\right)$ are positive values, and $\left(\frac{n-3}{2} - \frac{1}{2}\sqrt{n^2+2n-3}\right)$ and $\left(\frac{n-1}{2} - \frac{1}{2}\sqrt{n^2-2n+5}\right)$ are negative values for $n \geq 3$. Therefore $\max\{ i_+(A(B_n)), i_-(A(B_n))\} = 2n-2$ and hence, $bp(B_n) \geq 2(n-1)$.

Now, to show that $bp(B_n) = 2(n-1)$, let $V(K_n^1) = \{u_i : 1 \leq i \leq n\}$ and $V(K_n^2) = \{v_j : 1 \leq j \leq n\}$. Assume that $e = u_n v_1$ is the bridge from K_n^1 to K_n^2. By Graham and Pollak Theorem $bp(K_n^i) = n-1$ for each $i = 1, 2$. The edges of K_n^1 are covered by the disjoint stars generated by $\{u_2, u_3, ..., u_n\}$. The edges K_n^2 are covered by the disjoint stars generated by $\{v_2, v_3, ..., v_n\}$. Also the bridge e is covered by the star generated by the vertex u_n. So, $bp(B_n) \leq 2(n-1)$ and therefore, B_n is an eigensharp graph with $bp(B_n) = \max\{i_+(A(B_n)), i_-(A(B_n))\} = 2n-2$. □

Theorem 2.2. *The complement of the barbell graph is eigensharp.*

Proof. Assume that $V(K_n^1) = \{u_i : 1 \leq i \leq n\}$ and $V(K_n^2) = \{v_j : 1 \leq j \leq n\}$ are the sets whose union constitutes the set of vertex of the graph B_n. Suppose that $e = u_1v_1$ is the bridge connecting K_n^1 and K_n^2. Then, for the complement graph $\overline{B_n}$, for each $1 < i \leq n$ and $1 < j \leq n$, we have $u_iv_j \in E(\overline{B_n})$ except $e = u_1v_1 \notin E(\overline{B_n})$.
Since the subgraphs induced by $V(K_n^1)$ and $V(K_n^2)$ are null graphs in $\overline{B_n}$, we have $\overline{B_n} \simeq K_{n,n} - e$. Hence the adjacency matrix of $\overline{B_n}$ is $A = \begin{bmatrix} 0 & Q \\ Q & 0 \end{bmatrix}_{2n\times 2n}$, where Q is the $n \times n$ matrix with all entries are 1 except the entry $q_{1,1}$ is zero.
Now, let $B = \left(\begin{bmatrix} 1 & 1 & 1 & . & . & . & 1 \\ 1 & 2 & 2 & . & . & . & 2 \end{bmatrix}_{2\times n}\right)^t$, $C = \begin{bmatrix} -1 & 1 & 1 & . & . & . & 1 \\ 1 & 0 & 0 & . & . & . & 0 \end{bmatrix}_{2\times n}$ and let $D = CB = \begin{bmatrix} n-2 & 2n-3 \\ 1 & 1 \end{bmatrix}$. Then $Q = BC$ with spanning rank 2 and so, by [9], the characteristic polynomial of Q is $P_Q(x) = x^{n-2}P_D(x)$. Hence,

$$P_Q(x) = x^{n-2}(x^2 - (n-1)x - (n-1)),$$

and so, $\sigma(Q) = \begin{pmatrix} 0 & \frac{(n-1)\pm \sqrt[2]{n^2+2n-3}}{2} \\ n-2 & 1 \end{pmatrix}$.

Now, $\det(A - \lambda I) = \det(\lambda^2 I - Q^2)$; i.e., $P_A(x) = P_Q(x)P_{-Q}(x)$ [7]. Thus, we have

$$\sigma(\overline{B_n}) = \begin{pmatrix} 0 & \pm\frac{1}{2}\left((n-1) \pm \sqrt{n^2+2n-3}\right) \\ 2n-4 & 1 \end{pmatrix}$$

By simple calculations, we conclude that $n-1 < \sqrt{n^2+2n-3} = \sqrt{(n-1)(n+3)}$; i.e., $(n-1) - \sqrt{n^2+2n-3} < 0$ for $n \geq 2$.
Therefore, $\max\left\{i_+(A(\overline{B_n})), i_-(A(\overline{B_n}))\right\} = 2$ and so $bp(\overline{B_n}) \geq 2$.

On the other hand, since $\overline{B_n} \simeq K_{n,n} - e$, $\overline{B_n}$ have a biclique partition consists of the subgraph $K_{n-1,n}$ with vertex set $V = \{u_2, .., u_n, v_1, v_2, ..., v_n\}$ and the star S generated by the vertex u_1. Therefore, $bp(\overline{B_n}) = 2$. □

3 The Sun Graph and its Complement.

In this section, we study the biclique partition number of the sun graph $S(n)$ and its complement. Moreover, we show that $S(n)$ and $\overline{S(n)}$ are eigensharp

graphs for all $n \geq 2$.
The sun graph $S(n)$ is a graph consisting of $2n$ vertices obtained by attaching n pendant edges to a complete graph K_n. Let $V(S(n)) = \{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set of $S(n)$ where, $u_iu_j \in E(S(n))$ for each $1 \leq i, j \leq n$ and u_iv_i is a pendent edge in $S(n)$.

Theorem 3.1. *The Sun Graph is eigensharp.*

Proof. By the definition of $E(S(n))$, the adjacency matrix $A(S(n)) = \begin{bmatrix} A(K_n) & I_n \\ I_n & 0 \end{bmatrix}$, where $A(K_n)$ is the adjacency matrix of K_n and I_n is the identity matrix of size $n \times n$. So, by Lemma 1
$i_\pm \begin{bmatrix} A(K_n) & I_n \\ I_n & 0 \end{bmatrix} = i_\pm(A(K_n)) + i_\pm(-A(K_n^{-1})) = n$
which leads to $bp(S(n)) \geq \max\{n, n\} = n$.

On the other hand, let $W = \{S_1, ..., S_n\}$ be the set of n disjoint stars in $S(n)$ generated by $u_1, ..., u_n$ respectively. Then S_i contains at least the edge u_iv_i for each $1 \leq i \leq n$. Thus, W forms a biclique partition of $S(n)$ with minimum cardinality equals n. Hence, $bp\,(S(n)) = n$ and so $S(n)$ is an eigensharp graph. □

Note that, the corona product $K_n \odot K_1$ has been shown that it has the eigensharp property [6]. Since the sungraph $S(n)$ is isomorphic to $K_n \odot K_1$, this confirms that $S(n)$ is eigensharp.

Theorem 3.2. *The complement of the Sun Graph is eigensharp.*

Proof. Since the adjacency matrix of $\overline{S(n)}$ is $A(\overline{S(n)}) = \begin{bmatrix} A(K_n) & A(K_n) \\ A(K_n) & 0 \end{bmatrix}$, then by Lemma 1, $i_\pm \begin{bmatrix} A(K_n) & A(K_n) \\ A(K_n) & 0 \end{bmatrix} = i_\pm(A(K_n)) + i_\mp(A(K_n))$. Thus $\max\{i_+(A(\overline{S(n)})), i_-(A(\overline{S(n)}))\} = \max\{n, n\} = n$.

Now, let $H = \{S_1, S_2, ..., S_n\}$ be the set of n disjoint stars in $\overline{S(n)}$ generated by $v_1, ..., v_n$ respectively. Since the graph induced by $\{u_1, ..., u_n\}$ is a null subgraph, H forms a biclique partition of $\overline{S(n)}$ with minimum cardinality equals n. Thus, $bp(\overline{S(n)}) = n$ and so $\overline{S(n)}$ is an eigensharp graph. □

4 The Friendship Graph and its Complement

In this section, we study the biclique partition number of the friendship graph F_n and its complement $\overline{F_n}$. Moreover, we show that the graphs F_n and $\overline{F_n}$ are eigensharp graphs for all $n \geq 2$.

The friendship graph F_n is the joint of a one vertex v with disjoint n path graphs K_2. Let $V(F_n) = \{v, u_i : 1 \leq i \leq 2n\}$ be the vertex set of F_n. Then $E(F_n) = \{vu_i, 1 \leq i \leq 2n\} \cup \left\{u_{2j-1}u_{2j}, 1 \leq j \leq n\right\}$.

Theorem 4.1. *The Friendship graph is eigensharp for $n \geq 2$.*

Proof. Let H_1 be the subgraph of F_n with a one vertex v,; let H_2 be the graph induced by the disjoint union of the n path graphs K_2. Then $F_n = H_1 \vee H_2$ with

$$i_-(A(H_1)) = i_+(A(H_1)) = 0$$

and

$$A(H_2) = \begin{bmatrix} A(K_2) & 0 & . & . & . & 0 \\ 0 & A(K_2) & 0 & . & . & . \\ . & 0 & . & . & . & . \\ . & . & . & . & 0 & . \\ . & . & . & 0 & A(K_2) & 0 \\ 0 & 0 & . & . & 0 & A(K_2) \end{bmatrix}_{2n\times 2n}$$

thus, $i_-(A(H_2)) = i_+(A(H_2)) = n$. Since each of H_1 and H_2 are regular graphs, by [2], we have $i_-(A(F_n)) = n+1$ and $i_+(A(F_n)) = n-1$ and hence $bp(F_n) \geq n+1$. Now by the definition of $E(F_n)$, one can see that the set $B = \{vu_i, 1 \leq i \leq 2n\}$ is a star and the set $D^j = \{u_{2j-1}u_{2j}, 1 \leq j \leq n\}$ is a complete bipartite graph $K_{1,1}$, thus the set $\{B, D^1, D^2, ..., D^n\}$ is a biclique partition of F_n with minimum cardinality $n+1$. Hence, $bp\,(F_n) = n+1$, so F_n is an eigensharp graph. □

Theorem 4.2. *The complement of friendship graph is eigensharp for $n \geq 2$.*

Proof. Let v be the vertex in F_n join with disjoint n path graphs K_2 induced by the set of vertex $\{u_i : 1 \leq i \leq 2n\}$. Thus, $\overline{F_n}$, the complement of F_n, is decomposed by two components K_1 and W, where $K_1 = v$ and W is a graph of order $2n$ in which any pair of distinct verities u_iu_j forms an edge with the exception of the pairs $\left\{u_{2j-1}u_{2j}, 1 \leq j \leq n\right\}$. Therefore, $\sigma(\overline{F_n}) = \{0\} \cup \sigma(W)$, where the adjacency matrix of W is $A(W) = \begin{bmatrix} A(K_n) & A(K_n) \\ A(K_n) & A(K_n) \end{bmatrix}$. Thus, by

Lemma 1, $i_{\pm}(A(W)) = i_{\pm}(A(K_n))$; then

$$bp\left(\overline{F_n}\right) \geq \max\{i_+(A(K_n)), i_-(A(K_n))\} = n-1.$$

On the other hand, suppose that $X_1 = \{u_1, u_2\}$, $Y_1 = \{u_3, u_4, ..., u_{2n}\}$, $X_2 = \{u_3, u_4\}$, $Y_2 = \{u_5, u_6, ..., u_{2n}\}, ..., X_{n-1} = \{u_{2n-3}, u_{2n-2}\}$, $Y_{n-1} = \{u_{2n-1}, u_{2n}\}$. Let $K(X_s, Y_s)$ be the complete bipartite subgraph with two bipartite sets X_s and $Y_s, 1 \leq s \leq n-1$. Then $H = \{K(X_s, Y_s) : 1 \leq s \leq n-1\}$ is a set of biclique partition of a graph $\overline{F_n}$ such that $K(X_s, Y_s) \backsimeq K_{2,2n-2s}$. Since H is a biclique partition of $\overline{F_n}$ with cardinality $n-1$, $bp\left(\overline{F_n}\right) = n-1$ and so $\overline{F_n}$ is an eigensharp graph. □

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