International Journal of Mathematics and
Computer Science, **19**(2024), no. 2, 307–310

On T-Hopfcity of modules

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(Received August 1, 2023, Accepted September 6, 2023,
Published November 10, 2023)

Abstract

In this paper, we introduce a new concept called T-Hopfian modules, which is a generalization of the notion of Hopfian modules. We show that if a module M satisfies the ACC on non-T-small submodules, then it is T-Hopfian. We explore additional properties of T-Hopfian modules and provide examples for illustration.

1 Introduction

Throughout this paper, all rings have identity and all modules are unital right modules. The study of modules based on properties of their endomorphisms has been a topic of interest in mathematics. Hiremath introduced the concept of "Hopfian modules". An R-module M is Hopfian if every surjective endomorphism is an automorphism. Later, Varadarajan introduced "co-Hopfian modules." An R-module M is co-Hopfian if every injective endomorphism is an automorphism.

Building on the concept of Hopfian modules, a proper generalization called "generalized Hopfian modules" was introduced and studied in [7].

Key words and phrases: Hopfian modules, T-Hopfian modules, generalized Hopfian modules.
AMS (MOS) Subject Classifications: 16D10, 16D40, 16D90.
The corresponding author is Abderrahim El Moussaouy.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

Since their introduction, these concepts have sparked interest in various research works, and other generalizations and related topics have been explored and studied by different authors

In [1], the concept of T-small submodules is introduced as a generalization of small submodules within the context of R-modules. Here, a submodule K of an R-module M is said to be T-small in M, denoted by $K \ll_T M$, where T is a submodule of M. The definition of T-smallness involves the following condition: For every submodule L of M such that $T \subseteq K + L$, the condition $T \subseteq L$ holds.

In this paper, we propose a novel generalization of Hopfian modules, which we term T-Hopfian modules. Building upon previous works on Hopfian and generalized Hopfian modules, we introduce this new concept to explore module properties with a focus on specific submodules.

2 T-Hopfian Modules

Definition 2.1. *An R-module M is called "T-Hopfian" if every epimorphism of M has a T-small kernel.*

The following example demonstrates that Hopfian modules are a proper subclass of T-Hopfian modules. In other words, there exist modules that are T-Hopfian but not Hopfian.

Example 2.2. *In $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$, let $H_m =< 1/p^m + \mathbb{Z} >$ and $H_n =< 1/p^n + \mathbb{Z} >$. If $m > n$, then $H_n \ll_{H_m} \mathbb{Z}_{p^\infty}$, by [1, example 2.2(e)]. Hence every submodule of H_n is H_m-small and so H_n is H_m-Hopfian $\mathbb{Z}$-module. But the multiplication by a prime p induces a surjective endomorphism of H_n which is not an automorphism.*

Theorem 2.3. *For an R-module M and a submodule T of M, the following conditions are equivalent:*
(i) M is T-Hopfian.
(ii) There exists a fully invariant submodule N of M such that N is T-small in M and the quotient module M/N is T/N-Hopfian.

Proof. $(i) \Rightarrow (ii)$. This is evident by just taking $N = 0$.
$(ii) \Rightarrow (i)$. Given that N is a fully invariant T-small submodule of M, and, further, the quotient module M/N is T/N-Hopfian, the epimorphism $f : M \to M$ induces a well-defined epimorphism $g : M/N \to M/N$, defined by $g(m+N) = f(m)+N$. The fact that f is an epimorphism guarantees that g

is well-defined. Since M/N is T/N-Hopfian, it follows that $\ker g \ll_{T/N} M/N$. Suppose that $\ker g = L/N$, for some submodule L of M. Then $L/N \ll_{T/N} M/N$. Since $N \ll_T M$, by [1], $L \ll_T M$. As $Kerf$ is a submodule of L, we obtain $Kerf \ll_T M$. Therefore, M is T-Hopfian.

Proposition 2.4. *Let M be a T-Hopfian R-module, where T is a submodule of M. Consider an epimorphism f of M. If N is a T-small submodule of M, then $f^{-1}(N)$ is $f^{-1}(T)$-small in M.*

Proof. Let f be a surjective endomorphism of M and let N be a T-small submodule of M. If L is a submodule of M containing $Kerf$ with $f^{-1}(T)/Kerf \subseteq f^{-1}(N)/Kerf + L/Kerf$, then $f^{-1}(T) \subseteq f^{-1}(N) + L$. Hence $T \subseteq N + f(L)$. Since $N \ll_T M$, $T \subseteq f(L)$. As $Kerf \subseteq L$, $f^{-1}(T) \subseteq L$. Therefore, $f^{-1}(N)/kerf \ll_{f^{-1}(T)/Kerf} M/Kerf$. Now, $Kerf \ll_T M$ because M is T-Hopfian. By [1], $f^{-1}(N) \ll_{f^{-1}(T)} M$.

Proposition 2.5. *Let N be a fully invariant submodule of M such that the quotient module M/N is Hopfian and let T be a submodule of N. If N is T-Hopfian, then M is also T-Hopfian.*

Proof. Let $f : M \to M$ be a surjective endomorphism. Since the induced map $g : M/N \to M/N$ is epimorphism, it must be an automorphism because M/N is Hopfian. Hence $N = f^{-1}(N)$. Therefore, the restriction of f to N, denoted as $f/N : N \to N$, is also a surjective endomorphism. Now, if N is T-Hopfian, then $Kerf \cap N \ll_T N$. Since $Kerf$ is a submodule of N, $Kerf \ll_T N \leq M$. By [1], we have $Kerf \ll_T M$. Therefore, M is a T-Hopfian module.

Proposition 2.6. *Let an R-module M have a certain property $\mathfrak{P}$ that is preserved under isomorphism, and for every non-T-small submodule N of M, the quotient module M/N also has the property $\mathfrak{P}$, and further, M satisfies the Ascending Chain Condition (ACC) on such submodules N. Then M is T-Hopfian.*

Proof. Suppose, to get a contradiction, that M is not T-Hopfian. This means that there exists a submodule N_1 of M which is not T-small in M and the quotient module M/N_1 is isomorphic to M. Since M/N_1 is not T-Hopfian but satisfies the property $\mathfrak{P}$, there exists a submodule N_2 of M containing N_1 such that the quotient module N_2/N_1 is not T/N_1-small in M/N_1 and M/N_2 is isomorphic to M/N_1. Repeating this process, we obtain a chain of submodules $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n$ of M, where each N_i is not

T-small in M and M/N_i is isomorphic to M/N_1 for all i. This contradicts the assumption that M satisfies the ACC on non-T-small submodules, as the chain of submodules is strictly ascending. Therefore, the assumption that M is not T-Hopfian must be false and thus M is T-Hopfian.

Corollary 2.7. *Assume M is an R-module and T is a submodule of M. If M satisfies the ACC on non-T-small submodules, then M is a T-Hopfian module.*

Proof. Considering that M is nonzero with the ACC on non-T-small submodules and $\mathfrak{P}$ denotes the property of being nonzero, according to Proposition 2.6, M must be T-Hopfian.

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