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# Some properties of convex fuzzy normed space

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**Abstract**

The main goal here is to introduce another type of fuzzy normed space which we call a convex fuzzy normed space. Moreover, we give important properties of such a space.

## 1 Introduction

In 2012, Kider [1] defined new types of fuzzy normed spaces. Then, in 2017, Kider and Kadhum [2] introduced the fuzzy norm for a fuzzy bounded linear operator on a fuzzy normed space. Later, in 2018, Kider and Ali [3] introduced the notion of fuzzy absolute value and they investigated basic properties of the finite dimensional fuzzy normed space. In 2019, Gheeab and Kider [4] gave the definition of a general fuzzy normed space. In the same year, Kadhum and Kider [5] presented the notion of a fuzzy compact linear operator and studied its basic properties. In 2020, Kider and Khudhair [6] proved some properties of fuzzy compact a-fuzzy normed space and finite dimensional a-fuzzy normed space. In 2022, Khalaf and Kider [7] extended a

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linear operator on a-fuzzy normed space when it is fuzzy compact. In 2023, Abbas and Kider [8] showed basic properties of a-fuzzy normed algebra.

## 2 The convex fuzzy absolute value and its basic properties

**Definition 2.1.** *[8] If $\mathcal{U} \neq \emptyset$, then a fuzzy set $\mathcal{D}$: $\mathcal{U} \to$ [0,1] in $\mathcal{U}$ is a function with $0 \leq \mathcal{D}(u) \leq 1$ for all $u \in \mathcal{U}$.*

**Definition 2.2.** *Let $A_{\mathbb{R}}$: $\mathbb{R} \to$[0, 1] be a fuzzy set that satisfies:*
(1)$A_{\mathbb{R}}(\delta) \in [0, 1]$,
(2)$A_{\mathbb{R}}(\gamma) = 0 \Longleftrightarrow \gamma = 0$,
(3)$A_{\mathbb{R}}(\gamma).A_{\mathbb{R}}(\delta) \geq A_{\mathbb{R}}(\gamma\delta)$,
(4)$\sigma A_{\mathbb{R}}(\gamma) + \mu A_{\mathbb{R}}(\delta) \geq A_{\mathbb{R}}(\gamma + \delta)$,
*for all $\sigma$, $\mu \in$ [0, 1] with $\sigma + \mu$ =1 and for all $\gamma$, $\delta \in \mathbb{R}$.*
*Then ($\mathbb{R}$, $A_{\mathbb{R}}$) is called a convex fuzzy absolute value space or simply c-FAVS.*

**Definition 2.3.** *If ($\mathbb{R}$, $A_{\mathbb{R}}$) is a convex fuzzy absolute value space, then define $A_{\mathbb{R}}(\alpha)$= $A_{\mathbb{R}}(-\alpha)$ for all $\alpha \in \mathbb{R}$ and $A_{\mathbb{R}}$(1)=1.*

**Example 2.4.** *Define $A^{|\cdot|}$: $\mathbb{R} \longrightarrow [0, 1]$ by* $A^{|\cdot|}(\delta) = \begin{cases} \frac{1}{|\delta|} & if \ \ \delta \neq 0 \\ 0 & if \ \ \delta = 0 \end{cases}$
*for all $\delta \in$ $\mathbb{R}$. Then ($\mathbb{R}$, $A^{|\cdot|}$ ) is a convex fuzzy absolute value space.*

**Example 2.5.** *Let $A_{|.|}(\alpha) = \frac{|\alpha|}{1+|\alpha|}$ for all $\alpha \in \mathbb{R}$, where ($\mathbb{R}$,|.|) is the absolute value space. Then ($\mathbb{R}$, $A_{|.|}$ ) is a convex fuzzy absolute value space.*

**Definition 2.6.** *Let ($\mathbb{R}$, $A_{\mathbb{R}}$) be a convex fuzzy absolute value space and let $\{\alpha_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{R}$. Then $\{\alpha_k\}_{k=1}^{\infty}$ is convex fuzzy and approaches $\alpha \in \mathbb{R}$ when $k \to \infty$ if $\forall$ $\sigma \in$ (0,1), $\exists$ $\mathcal{N} \in \mathbb{N}$ such that $A_{\mathbb{R}}(\alpha_k - \alpha) < \sigma$, for all k$\geq$ $\mathcal{N}$. We write $\lim_{k\to\infty} \alpha_k = \alpha$ or $\alpha_k \to \alpha$ as $k \to \infty$ or $\lim_{k\to\infty} A_{\mathbb{R}}(\alpha_k - \alpha) = 0$.*

**Theorem 2.7.** *Let ($\mathbb{R}$, $A_{\mathbb{R}}$) be a convex fuzzy absolute value space and let $\{\alpha_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{R}$ such that $\alpha_k \to \alpha$ and $\alpha_k \to \beta$ as $k \to \infty$. Then $\alpha$=$\beta$.*

**Definition 2.8.** *Let ($\mathbb{R}$, $A_{\mathbb{R}}$) be a convex fuzzy absolute value space and let $\{\alpha_k\}_{k=1}^{\infty} \in \mathbb{R}$, then $\{\alpha_k\}_{k=1}^{\infty}$ is a convex fuzzy Cauchy sequence in $\mathbb{R}$ if $\forall$ $\sigma \in$ (0,1), $\exists$ $\mathcal{N} \in \mathbb{N}$ such that $A_{\mathbb{R}}(\alpha_n - \alpha_m) < \sigma$, $\forall$ n, $m \geq \mathcal{N}$.*

**Theorem 2.9.** *Let* ($\mathbb{R}$, $A_{\mathbb{R}}$) *be the convex fuzzy absolute value space and let* $\{\alpha_k\}_{k=1}^{\infty}$*be a sequence in* $\mathbb{R}$ *such that* $\alpha_k \to \alpha$ *as* $k \to \infty$, *then it is convex fuzzy Cauchy.*

**Theorem 2.10.** *Let* ($\mathbb{R}$, $A_{\mathbb{R}}$) *be a convex fuzzy absolute value space and let* $\{\alpha_k\}_{k=1}^{\infty}$ *be a sequence in* $\mathbb{R}$ *such that* $\alpha_k \to \alpha$ *then all* ($\alpha_{k_n}$) $\subseteq \{\alpha_k\}_{k=1}^{\infty}$ *satisfying* $\alpha_{k_n} \to \alpha$.

**Definition 2.11.** *Let* ($\mathbb{R}$, $A_{\mathbb{R}}$) *be convex fuzzy absolute value space. The sequence* $\{\sigma_k\}_{k=1}^{\infty}$ *in* $\mathbb{R}$ *is said to be convex fuzzy bounded if there exists* $\alpha \in$ *[0,1] such that* $A_{\mathbb{R}}(\sigma_k) < \alpha$, *for all* $k \in \mathbb{N}$.

**Theorem 2.12.** *Let* ($\mathbb{R}$, $A_{\mathbb{R}}$) *be a convex fuzzy absolute value space. If the sequence* $\{\sigma_k\}_{k=1}^{\infty}$ *in* $\mathbb{R}$ *is convex fuzzy approaches to the limit* $\sigma$. *Then it is convex fuzzy bounded.*

*Proof.* Suppose that $\{\sigma_k\}_{k=1}^{\infty}$ in $\mathbb{R}$ is convex fuzzy and approaches the limit $\sigma$ as $k \to \infty$. Then for every $\alpha \in (0,\ 1)$ there exists $N \in \mathbb{N}$ such that $A_{\mathbb{R}}(\sigma_k - \sigma) < \alpha$, for all $k \geq N$. This implies that $A_{\mathbb{R}}(\sigma_k) = A_{\mathbb{R}}(\sigma + \sigma_k - \sigma) \leq \gamma A_{\mathbb{R}}(\sigma) + \delta A_{\mathbb{R}}(\sigma_k - \sigma) < \gamma A_{\mathbb{R}}(\sigma) + \delta\alpha$, where $\gamma + \delta$=1. Now put $\theta = A_{\mathbb{R}}(\sigma)$, for some $\theta \in [0,1]$. Then $A_{\mathbb{R}}(\sigma_k) < \gamma\theta + \delta\alpha$. Choose $\mu \in [0,1]$ with $\gamma\theta + \delta\alpha < \mu$. Hence $A_{\mathbb{R}}(\sigma_k) < \mu$, for each $k \in \mathbb{N}$. Thus $\{\sigma_k\}_{k=1}^{\infty}$ is convex fuzzy bounded. □

**Theorem 2.13.** *Let* ($\mathbb{R}$, $A_{\mathbb{R}}$) *be a convex fuzzy absolute value space and let* $\{\sigma_k\}_{k=1}^{\infty}$, $\{\theta_k\}_{k=1}^{\infty}$ *be two sequences in* $\mathbb{R}$. *If* $\{\sigma_k\}_{k=1}^{\infty}$ *is convex fuzzy approaches* $\sigma$ *as* $k \to \infty$ *and* $\{\theta_k\}_{k=1}^{\infty}$ *is convex fuzzy and approaches* $\theta$ *as* $k \to \infty$. *Then*
*(1)* $\{\sigma_k + \theta_k\}_{k=1}^{\infty}$ *is convex fuzzy and approaches* $\sigma + \theta$.
*(2)* $\{\beta\sigma_k\}_{k=1}^{\infty}$ *is convex fuzzy and approaches* $\beta\sigma$, *for any* $0 \neq \beta \in \mathbb{R}$.

*Proof.* The proof is clear and hence is omitted. □

**Theorem 2.14.** *Let* ($\mathbb{R}$, $A_{\mathbb{R}}$) *be a convex fuzzy absolute value space and let* $\{\sigma_k\}_{k=1}^{\infty}$, $\{\theta_k\}_{k=1}^{\infty}$ *be two sequences in* $\mathbb{R}$. *If* $\{\sigma_k\}_{k=1}^{\infty}$ *is convex fuzzy and approaches* $\sigma$ *as* $k \to \infty$ *and* $\{\theta_k\}_{k=1}^{\infty}$ *is convex fuzzy and approaches* $\theta$ *as* $k \to \infty$. *Then* $\{\sigma_k\theta_k\}_{k=1}^{\infty}$ *is convex fuzzy and approaches* $\sigma\theta$ *as* $k \to \infty$.

*Proof.* Since $\{\sigma_k\}_{k=1}^{\infty}$ is convex fuzzy approaches the limit $\sigma$, for every $\alpha \in (0,1)$, there exists $N_1 \in \mathbb{N}$ such that $A_{\mathbb{R}}(\sigma_k - \sigma) < \alpha$ for all $k \geq N_1$. Also, since $\{\theta_k\}_{k=1}^{\infty}$ is fuzzy and approaches the limit $\theta$, for every $\varepsilon \in (0,1)$, there exists $N_2 \in \mathbb{N}$ such that $A_{\mathbb{R}}(\theta_k - \theta) < \varepsilon$, for all $k \geq N_2$. Now, choose $N = min\{N_1, N_2\}$. Then, for each $k \geq N$, we have

$A_{\mathbb{R}}(\sigma_k\theta_k-\sigma\theta) = A_{\mathbb{R}}(\sigma_k\theta_k-\sigma\theta_k+\sigma\theta_k- \sigma\theta) \le \gamma A_{\mathbb{R}}(\sigma_k\theta_k-\sigma\theta_k)+\delta A_{\mathbb{R}}(\sigma\theta_k-\sigma\theta)$, where $\gamma+\delta=1$. Now, $A_{\mathbb{R}}(\sigma_k\theta_k-\sigma\theta) \le \gamma A_{\mathbb{R}}[(\sigma_k-\sigma)\theta_k]+\delta A_{\mathbb{R}}[\sigma(\theta_k-\theta)]$ $\le \gamma A_{\mathbb{R}}(\sigma_k-\sigma)A_{\mathbb{R}}(\theta_k)+\delta A_{\mathbb{R}}(\sigma)A_{\mathbb{R}}(\theta_k-\theta) \le \gamma\sigma A_{\mathbb{R}}(\theta_k)+\delta A_{\mathbb{R}}(\sigma)\ \theta$. Put $A_{\mathbb{R}}(\theta_k)=\rho$ and $A_{\mathbb{R}}(\sigma)=\tau$ and choose $\varepsilon \in (0,1)$ such that $(\gamma\ \sigma\rho+\delta\ \tau\ \theta)<\varepsilon$. Thus $A_{\mathbb{R}}(\sigma_k\theta_k-\sigma\theta)<\varepsilon$, for each $k \ge N$.
Hence $\{\sigma_k\theta_k\}_{k=1}^{\infty}$ is convex fuzzy and approaches the limit $\sigma\theta$ as $k \to \infty$. □

# 3 Properties of Convex Fuzzy Normed Space

**Definition 3.1.** *Let $\mathcal{U}$ be a vector space over $\mathbb{R}$, let $(\mathbb{R}, A_{\mathbb{R}})$ be a convex fuzzy absolute value space, and let $\mathfrak{N}$: $\mathcal{U} \to I$ be a fuzzy set. If $\mathfrak{N}$ satisfies:*
*(1) $0 \le \mathfrak{N}(u) \le 1$,*
*(2) $\mathfrak{N}(u)=0$ if and only if $u=0$,*
*(3) $\mathfrak{N}(\alpha u) \le A_{\mathbb{R}}(\alpha)\ \mathfrak{N}(u)$ for all $0 \ne \alpha \in \mathbb{R}$,*
*(4) $\mathfrak{N}(u+v) \le \gamma\mathfrak{N}(u)+\delta\mathfrak{N}(v)$,*
*where $\gamma+\delta=1$, for all $u, v \in \mathcal{U}$. Then $(\mathcal{U}, \mathfrak{N})$ is called a convex fuzzy normed space.*

**Remark 3.2.** *(1) If $\delta, \sigma \in [0,1]$, then $(\alpha\delta+(1-\alpha)\sigma) \in [0,1]$, for any $\alpha \in [0,1]$. In general, if $\sigma_1, \sigma_2, \ldots, \sigma_k \in [0,1]$, then $(\alpha_1\sigma_1+\alpha_2\sigma_2+\ldots+\alpha_k\sigma_k) \in [0,1]$ for any $\alpha_1, \alpha_2, \ldots, \alpha_k \in [0,1]$ with $\alpha_1+\alpha_2+\ldots+\alpha_k=1$.*

*Proof.* Since $\alpha\delta \le \alpha$ and $(1-\alpha)\sigma \le (1-\alpha)$, $(\alpha\delta+(1-\alpha)\sigma) \le \alpha+(1-\alpha)=1$. Similarly, since $\alpha_1\sigma_1 \le \alpha_1$, $\alpha_2\sigma_2 \le \alpha_2$, ..., $\alpha_k\sigma_k \le \alpha_k$, $(\alpha_1\sigma_1+\alpha_2\sigma_2+\ldots+\alpha_k\sigma_k) \le \alpha_1+\alpha_2+\ldots+\alpha_k=1$.
(2) By induction, it is clear that $\mathfrak{N}(u_1+u_2+\ldots+u_k) \le \delta_1\mathfrak{N}_1(u_1)+\delta_2\mathfrak{N}_2(u_2)+,\ldots,+\delta_k\mathfrak{N}_k(u_k)$ for all $u_1, u_2, \ldots, u_k \in \mathcal{U}$ where $\delta_1+\delta_2+\ldots+\delta_3=1$. □

**Example 3.3.** *Define $\mathfrak{N}^{\|\cdot\|}$:$\mathcal{U} \to I$ by* $\mathfrak{N}^{\|\cdot\|}(u) = \begin{cases} \frac{1}{\|u\|} & if\ u \ne 0 \\ 0 & if\ \ u=0 \end{cases}$
*for all $u \in \mathcal{U}$. Then $(\mathcal{U}, \mathfrak{N}^{\|\cdot\|})$ is convex fuzzy normed space and is called the convex fuzzy normed space induced by $\|.\|$.*

*Proof.* We show that all conditions of Definition 3.1 are satisfied.
(1) It is clear that $\mathfrak{N}^{\|\cdot\|}(u) \in I$.
(2) $\mathfrak{N}^{\|\cdot\|}(u)=0$ if and only if $u=0$ follows immediately from the definition of $\mathfrak{N}^{\|\cdot\|}$.
(3) $A_{\mathbb{R}}(\alpha)\cdot\mathfrak{N}^{\|\cdot\|}(u) = \frac{1}{|\alpha|}\frac{1}{\|u\|} = \frac{1}{\|\alpha u\|} = \mathfrak{N}^{\|\cdot\|}(\alpha u)$.
(4) $\gamma\ \mathfrak{N}^{\|\cdot\|}(u)+\delta\mathfrak{N}^{\|\cdot\|}(v) = \frac{\gamma}{\|u\|}+\frac{\delta}{\|v\|} = \frac{\gamma\|v\|+\delta\|u\|}{\|u\|\|v\|} \ge \frac{1}{\|u+v\|} = \mathfrak{N}^{\|\cdot\|}(u+v)$.
Hence $(\mathcal{U}, \mathfrak{N}^{\|\cdot\|})$ is a convex fuzzy normed space when $\gamma+\delta=1$. □

**Theorem 3.4.** *If we define* $\mathfrak{N}$*:* $\mathbb{R} \rightarrow I$ *by* $\mathfrak{N}(\alpha)= A_{\mathbb{R}}(\alpha,)$ *for all* $\alpha \in \mathbb{R}$, *then* $(\mathbb{R}, \mathfrak{N})$ *is a convex fuzzy normed space.*

*Proof.* We show that all conditions of Definition 3.1 are satisfied:
(1) It is clear that $\mathfrak{N}(\alpha) \in I$, for all $\alpha \in \mathbb{R}$.
(2) $\mathfrak{N}(\alpha)$=0 if and only if $A_{\mathbb{R}}(\alpha)$=0 if and only if $\alpha$ =0.
(3) $\mathfrak{N}(\alpha\beta) = A_{\mathbb{R}}(\alpha\beta) \leq A_{\mathbb{R}}(\alpha)\ .\ A_{\mathbb{R}}(\beta) = \mathfrak{N}(\alpha)\ \mathfrak{N}(\beta)$ for all $0 \neq \alpha \in \mathbb{R}$.
(4) $\mathfrak{N}(\alpha+\beta) = A_{\mathbb{R}}(\alpha+\beta) \leq \gamma A_{\mathbb{R}}(\alpha) + \delta A_{\mathbb{R}}(\beta) = \gamma\mathfrak{N}(\alpha) + \delta\mathfrak{N}(\beta)$.
where $\gamma + \delta$=1. Hence $(\mathbb{R}, \mathfrak{N})$ is a convex fuzzy normed space. □

**Example 3.5.** *Let* $\mathcal{U} = C[a,b]$. *Let* $\mathfrak{N}(g) = max_{\alpha\in[a,\ b]}\ A_{\mathbb{R}}[g(\alpha)]$, *for all* $g \in \mathcal{U}$. *Then* $(\mathcal{U}, \mathfrak{N})$ *is a convex fuzzy normed space.*

*Proof.* We show that all conditions of Definition 3.1 are satisfied:
(1) It is clear that $\mathfrak{N}(g) \in$I for all $g \in \mathcal{U}$.
(2) $\mathfrak{N}(g) = 0$ if and only if $max_{\alpha\in[a,\ b]}\ A_{\mathbb{R}}[g(\alpha)]= 0$ if and only if $A_{\mathbb{R}}[g(\alpha)] = 0$ for all $\alpha \in$ [a, b] if and only if $g(\alpha)$=0, for all $\alpha \in$ [a, b] if and only if $g$ =0.
(3) $\mathfrak{N}(\sigma g) = max_{\alpha\in[a,\ b]} A_{\mathbb{R}}[\sigma g(\alpha)] \leq A_{\mathbb{R}}(\sigma).max_{\alpha\in[a,\ b]} A_{\mathbb{R}}[g(\alpha)] = A_{\mathbb{R}}(\sigma).\mathfrak{N}(g)$, for all $0 \neq \sigma \in \mathbb{R}$.
(4) $\frac{N}{g+k} = \max_{\alpha\in[a,b]} A_{\mathbb{R}}[(g+k)(\alpha)] = \max_{\alpha\in[a,b]} A_{\mathbb{R}}[(g(\alpha)+k(\alpha)] \leq max_{\alpha\in[a,\ b]}\gamma A_{\mathbb{R}}[(g(\alpha)]+ max_{\alpha\in[a,\ b]}\delta A_{\mathbb{R}}[(k(\alpha)]$, where $\gamma+\delta$=1. Now, $\mathfrak{N}(g+k)\ \leq \gamma max_{\alpha\in[a,\ b]} A_{\mathbb{R}}[(g(\alpha)]+ \delta max_{\alpha\in[a,\ b]} A_{\mathbb{R}}[(k(\alpha)] = \gamma\ \mathfrak{N}(g) + \delta\mathfrak{N}(k)$. Hence $(\mathcal{U}, \mathfrak{N})$ is a convex fuzzy normed space. □

**Example 3.6.** *Define* $\mathfrak{N}_{\|.\|}(u) = \frac{\|u\|}{1+\|u\|}$, *for all* $u \in \mathcal{U}$ *when* $(\mathcal{U}, \|.\|)$ *is a normed space. Then* $(\mathcal{U}, \mathfrak{N}_{\|.\|})$ *is a convex fuzzy normed space.*

*Proof.* To prove that $\mathfrak{N}_{\|.\|}$ satisfies all the conditions of Definition 3.1,
(1) $0 \leq \mathfrak{N}_{\|.\|}(u) \leq 1$ for all $u \in \mathcal{U}$;
(2) $u = 0 \Longleftrightarrow \|u\| = 0 \Longleftrightarrow \frac{\|u\|}{1+\|u\|} = 0 \Longleftrightarrow \mathfrak{N}_{\|.\|}(u) = 0$;
(3) $\mathfrak{N}_{\|.\|}(\sigma u) = \frac{\|\sigma u\|}{1+\|\sigma u\|} = \frac{|\sigma|\|u\|}{1+|\sigma|\|u\|} \leq [\frac{|\sigma|}{1+|\sigma|}].[\ \frac{\|u\|}{1+\|u\|}] =\ A_{|.|}(\sigma).\ \mathfrak{N}_{\|.\|}(u)$;
(4) To prove $\mathfrak{N}_{\|.\|}(u+ v) \leq \alpha\mathfrak{N}_{\|.\|}(u) + \beta\mathfrak{N}_{\|.\|}(v)$, for all $u$, $v$, $\in \mathcal{U}$, put $\beta$=$(1-\alpha)$. Then
$\mathfrak{N}_{\|.\|}(u+v) = 1 - \frac{1}{1+\|u+\ v\|} \leq 1 - \frac{1}{1+\alpha\|u\|+(1-\alpha)\|v\|} \leq \frac{\alpha\|u\|+(1-\alpha)\|v\|}{1+\alpha\|u\|+(1-\alpha)\|v\|} \leq \alpha\mathfrak{N}_{\|.\|}(u) + \beta\mathfrak{N}_{\|.\|}(v)$. Thus $(\mathcal{U}, \mathfrak{N}_{\|.\|})$ is a convex fuzzy normed space. □

**Theorem 3.7.** *Let* $(\mathcal{U}_1, \mathfrak{N}_1)$ *and* $(\mathcal{U}_2, \mathfrak{N}_2)$ *be convex fuzzy normed spaces and let* $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$. *Then* $(\mathcal{U}, \mathfrak{N})$ *is a convex fuzzy normed space, where* $\mathfrak{N}[(u_1, u_2)] = \gamma\mathfrak{N}_1(u_1) + \delta\mathfrak{N}_2(u_2)$, *for all* $(u_1, u_2) \in \mathcal{U}$ *where* $\gamma + \delta$*=1.*

*Proof.* We show that all conditions of Definition 3.1 are satisfied.
(1) Since $0\leq \mathfrak{N}_1(u_1) \leq 1$ and $0\leq \mathfrak{N}_2(u_2) \leq 1$, $0\leq \mathfrak{N}[(u_1,\, u_2)] \leq 1$
(2) $\mathfrak{N}[(u_1,\, u_2)]$=0 if and only if $\gamma\mathfrak{N}_1(u_1) + \delta\mathfrak{N}_2(u_2)$=0 if and only if $\mathfrak{N}_1(u_1)$=0 and $\mathfrak{N}_2(u_2)$=0 if and only if $u_1$=0 and $u_2$=0 if and only if $(u_1, u_2) = (0, 0)$
(3) $\mathfrak{N}[\alpha(u_1,\, u_2)]$= n$[(\alpha u_1,\, \alpha u_2)] = \gamma\mathfrak{N}_1(\alpha u_1) + \delta\mathfrak{N}_2(\alpha u_2)\leq A_{\mathbb{R}}(\alpha).\, \gamma\mathfrak{N}_1(\alpha u_1)+ A_{\mathbb{R}}(\alpha).\, \delta\mathfrak{N}_2(u_2) \;\leq A_{\mathbb{R}}(\alpha)\; [\gamma\mathfrak{N}_1(u_1) + \delta\mathfrak{N}_2(u_2)]$=a(c). n$[(u_1,\, u_2)]$
(4) $\mathfrak{N}[(u_1, u_2)+(v_1, v_2)] = \mathfrak{N}[(u_1+v_1)+(u_2+v_2)] = \gamma\mathfrak{N}_1(u_1+v_1)+\delta\mathfrak{N}_2(u_2+v_2) \leq \gamma[\sigma\mathfrak{N}_1(u_1) + \theta\mathfrak{N}_1(v_1)] + \delta[\sigma\mathfrak{N}_2(u_2) + \theta\mathfrak{N}_2(v_2)]$,
where $\gamma+\delta$=1 and $\sigma+\theta$=1.
Now, $\mathfrak{N}[(u_1, u_2)+(v_1, v_2)] \leq \sigma[\gamma\mathfrak{N}_1(u_1)+\delta\mathfrak{N}_2(u_2)]+\theta[\gamma\mathfrak{N}_1(v_1)+\delta\mathfrak{N}_2(v_2)] = \sigma\mathfrak{N}[(u_1, u_2)]+\theta\mathfrak{N}[(v_1, v_2)]$. Hence $(\mathcal{U}, \mathfrak{N})$ is a convex fuzzy normed space. □

Now, the following corollaries follow easily.

**Corollary 3.8.** *If* $(\mathcal{U}_1, \mathfrak{N}_1)$, $(\mathcal{U}_2, \mathfrak{N}_2)$, $\ldots$, $(\mathcal{U}_k, \mathfrak{N}_k)$ *are algebra fuzzy normed spaces, then* $(\mathcal{U}, \mathfrak{N})$ *is convex fuzzy normed space, where* $\mathcal{U}=\mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_k$ *and* $\mathfrak{N}[(u_1,\, u_2,\, \ldots,\, u_k)]= \delta_1\mathfrak{N}_1(u_1) +\delta_2\, \mathfrak{N}_2(u_2) + \ldots +\delta_k\, \mathfrak{N}_k(u_k)$, *for all* $(u_1,\, u_2,\, \ldots, u_k) \in \mathcal{U}$ *where* $\delta_1+\, \delta_2+\, \ldots.+\delta_3$*=1.*

**Corollary 3.9.** *If* $(\mathcal{U}, \mathfrak{N})$ *is a convex fuzzy normed space, then* $(\mathcal{U}^k, \mathfrak{N}_{\mathcal{U}})$ *is a convex fuzzy normed space, where* $\mathcal{U}^k= \mathcal{U}\times\, \mathcal{U}\times\ldots.\times\, \mathcal{U}$ *(k-times) and* $\mathfrak{N}_{\mathcal{U}}[(u_1,\, u_2,\, \ldots, u_k)]= \delta_1\mathfrak{N}(u_1) +\delta_2\, \mathfrak{N}(u_2) + \ldots +\delta_k\, \mathfrak{N}(u_k)$ *for all* $(u_1,\, u_2,\, \ldots, u_k) \in \mathcal{U}$, *where* $\delta_1+\, \delta_2+\, \ldots.+\delta_k$*=1.*

**Definition 3.10.** *Let* $(\mathcal{U}, \mathfrak{N})$ *be a convex fuzzy normed space and let* $(u_k)$ *be a sequence in* $\mathcal{U}$. *We say that* $(u_k)$ *is convex fuzzy convergent to the limit* $u$ *as* $k$ *approaches to* $\infty$ *if, for every* $\alpha \in (0, 1)$, *there exists* $N \in \mathbb{N}$ *such that* $\mathfrak{N}(\mathrm{u_k}-u) < \alpha$, *for all* $k \geq N$. *If* $(u_k)$ *is convex fuzzy convergent to the limit* $u$, *we write* $\lim_{\mathrm{k}\to\infty} \mathrm{u_k} = u$ *or* $\mathrm{u_k}\to u$ *as* $k \to infty$ *or* $\lim_{\mathrm{n}\to\infty} \mathfrak{N}(\mathrm{u_k}-u) = 0$.

**Definition 3.11.** *Suppose that* $(\mathcal{U}, \mathfrak{N})$ *is a convex fuzzy normed space. Put* $cfb(u,\, \alpha)= \{\, v \in \mathcal{U}\colon\, \mathfrak{N}(u-v) < \alpha\, \}$ *and* $cfb[u,\, \alpha]= \{\, v \in \mathcal{U}\colon\, \mathfrak{N}(u-v) < \alpha\}$. *Then* $cfb(u,\, \alpha)$ *and* $cfb[u,\, \alpha]$ *is called convex open and convex closed fuzzy ball with the center* $u \in \mathcal{U}$ *and radius* $\alpha$, *with* $\alpha \in (0, 1)$.

**Definition 3.12.** *Let* $(\mathcal{U}, \mathfrak{N})$ *be a convex fuzzy normed space and let* $(u_k)$ *be a sequence in* $\mathcal{U}$. *We say that* $(u_k)$ *is a convex fuzzy Cauchy sequence in* $\mathcal{U}$ *if, for every* $\varepsilon \in (0, 1)$, *there exists* $N \in \mathbb{N}$ *such that* $\mathfrak{N}(\mathrm{u_k}-\mathrm{u_m}) < \varepsilon$, *for all* $k, m \geq N$.

**Lemma 3.13.** *Let* $(\mathcal{U},\mathfrak{N})$ *be a convex fuzzy normed space and define* $A_{\mathbb{R}}[\mathfrak{N}\,(u)] = \mathfrak{N}\,(u)$, *for all* $u,\, v \in \mathcal{U}$. *Then* $A_{\mathbb{R}}[\mathfrak{N}\,(u)-\mathfrak{N}(v)] \leq \mathfrak{N}(u-v)$, *for all* $u, v \in \mathcal{U}$.

**Lemma 3.14.** *If ($\mathcal{U}$, $\mathfrak{N}$) is a convex fuzzy normed space, then $\mathfrak{N}(u-v) = \mathfrak{N}\ (v-u)$, for all $u, v \in \mathcal{U}$.*

*Proof.* $\mathfrak{N}(u-v) = \mathfrak{N}[(-1)(v-u)] \leq A_{\mathbb{R}}(-1)$. $\mathfrak{N}\ (v-u) = A_{\mathbb{R}}(1)$. $\mathfrak{N}\ (v-u) = \mathfrak{N}\ (v-u)$; that is, $\mathfrak{N}(u-v) \leq \mathfrak{N}\ (v-u)$. Similarly, $\mathfrak{N}\ (v-u) \leq \mathfrak{N}(u-v)$, from these two inequalities we conclude that $\mathfrak{N}\ (u-v) = \mathfrak{N}\ (v-u)$, for all $u, v \in \mathcal{U}$. □

**Definition 3.15.** *Let $\mathcal{U} \neq \emptyset$ and if the fuzzy set $\mathfrak{M}$: $\mathcal{U} \times \mathcal{U} \rightarrow [0, 1]$ satisfies*
*(1) $\mathfrak{W}(u, v) \in (0, 1)$,*
*(2) $u = v \Longleftrightarrow \mathfrak{M}(u, v) = 0$, (3) $\mathfrak{M}(u, v) = \mathfrak{M}(v, u)$,*
*(4) $\alpha\mathfrak{M}(u, w) + \beta\mathfrak{M}(w, v) \geq \mathfrak{M}(u, v)$, for all $\alpha$, $\beta \in [0, 1]$ with $\alpha + \beta = 1$ and for all $u$, $v$, $w \in \mathcal{U}$.*
*Then ($\mathcal{U}$, $\mathfrak{M}$) is a convex fuzzy metric space or a c-fuzzy metric space or simply a c-FMS.*

*The proof of the next result is clear and hence is omitted.*

**Theorem 3.16.** *If ($\mathcal{U}$, $\mathfrak{N}$) is a convex fuzzy normed space, then ($\mathcal{U}$, $\mathfrak{M}_{\mathfrak{N}}$) is a convex fuzzy metric space, where $\mathfrak{M}_{\mathfrak{N}}(u, v) = \mathfrak{N}(u-v)$, for all $u$, $v \in \mathcal{U}$. We call ($\mathcal{U}$, $\mathfrak{M}_{\mathfrak{N}}$) the convex fuzzy metric space induced by $\mathfrak{N}$.*

**Definition 3.17.** *If ($\mathcal{U}$, $\mathfrak{N}$) is a convex fuzzy normed space, then $\mathcal{W} \subseteq \mathcal{U}$ is known as convex Fuzzy open if cfb($w$,$\alpha$ ) $\subseteq \mathcal{W}$, for any arbitrary $w \in \mathcal{W}$ and for some $\alpha \in (0, 1)$. Also, $\mathcal{D} \subseteq \mathcal{U}$ is known as convex fuzzy closed if $\mathcal{D}^C$ is fuzzy open. Moreover, the convex fuzzy closure $\overline{\mathcal{D}}$ of $\mathcal{D}$, is defined to be the smallest fuzzy closed set containing $\mathcal{D}$. Also, $\mathcal{D} \subseteq \mathcal{U}$ is known as convex fuzzy dense in $\mathcal{U}$ if whenever $\overline{\mathcal{D}} = \mathcal{U}$.*

**Theorem 3.18.** *If cfb($u$, $\alpha$) is a convex open fuzzy ball in a convex fuzzy normed space ($\mathcal{U}$, $\mathfrak{N}$), then it is a convex fuzzy open set.*

*Proof.* Let cfb($u$, $\alpha$) be a convex open fuzzy ball, where $u \in \mathcal{U}$ and $\alpha \in (0, 1)$. Let $v \in$ cfb($u$, $\alpha$). So $\mathfrak{N}(u-v) < \alpha$. Let $\beta = \mathfrak{N}(u-v)$. So $\beta < \alpha$. Then there is $\sigma \in (0, 1)$ such that $(\gamma\ \beta + \delta\ \sigma\ ) < \alpha$. Now, consider the convex open fuzzy ball cfb($v$, $\sigma$). We show that cfb($v$, $\sigma$) $\subseteq$ cfb($u$, $\alpha$). Let $z \in$ cfb($v$, $\sigma$). So $\mathfrak{N}(v-z) < \sigma$. Hence $\mathfrak{N}(u-z) \leq \gamma\ \mathfrak{N}(u-v) + \delta\ \mathfrak{N}(v-z)$, where $\gamma + \delta = 1$. Now, $\mathfrak{N}(u-z) \leq \gamma\ \beta + \delta\ \sigma$ and so $z \in$ cfb($u$, $\alpha$); that is, cfb($v$, $\sigma$) $\subseteq$ cfb($u$, $\alpha$). Therefore, cfb($u$, $\alpha$) is a convex fuzzy open set. □

**Theorem 3.19.** *Let ($\mathcal{U}$, $\mathfrak{N}$) be a convex fuzzy normed space.*
*(1) If { $\mathcal{U}_{\mathrm{i}}$: $\mathrm{i} \in \mathrm{I}$ } is a family of convex fuzzy open sets, then $\cup_{\mathrm{i} \in \mathrm{I}} \mathcal{U}_{\mathrm{i}}$ is fuzzy*

*open set.*
*(2) If $\mathcal{U}_1$, $\mathcal{U}_2$,..., $\mathcal{U}_k$ represent a finite number of convex fuzzy open set, then $\cap_{i=1}^{k}\mathcal{U}_i$ is fuzzy open set.*

*Proof.*
(1) Suppose that { $\mathcal{U}_i$: i∈ $I$ } is a family of convex fuzzy open sets with $\mathcal{U} = \cup_{i\in I}\mathcal{U}_i$. Let $U \in \mathcal{U}$. Then $U \in \cup_{i\in I}\mathcal{U}_i$. So $w \in \mathcal{U}_i$, for some $i \in I$. Since $\mathcal{U}_i$ is a convex fuzzy open set, $\exists$ 0< $\sigma < 1$ satisfying cfb($U$, $\sigma$) $\subset$ $\mathcal{U}_i$. Thus cfb($U$, $\sigma$) $\subset$ $\mathcal{U}_i$ $\subseteq \cup_{i\in I}\mathcal{U}_i = \mathcal{W}$; that is, $\mathcal{U}$ is a convex fuzzy open set.
(2) Let $\mathcal{U}_1$, $\mathcal{U}_2$,..., $\mathcal{U}_k$ be convex fuzzy open sets and put $\mathcal{V} = \cap_{i=1}^{k}\mathcal{U}_i$. We show that $\mathcal{V}$ is a convex fuzzy open set. Let $v \in \mathcal{V}$. Then $v \in \mathcal{U}_i$, for each $1 \leq$ i $\leq k$. Hence $\exists$ $0 \leq \alpha_i \leq 1$ satisfying cfb($v$, $\alpha_i$) $\subset \mathcal{U}_i$ since $\mathcal{U}_i$ is a convex fuzzy open $\forall i = 1, 2, \ldots, k$. Let $\alpha = \min\{ \alpha_i : 1 \leq i \leq k \}$. So $\alpha \leq \alpha_i$, $\forall$ $1 \leq i \leq k$. So cfb($v$, $\alpha$) $\subset \mathcal{U}_i$ for all $1 \leq i \leq k$. Therefore, cfb($v$, $\alpha$) $\subseteq \cap_{i=1}^{k}\mathcal{U}_i = \mathcal{V}$, thus $\mathcal{V}$ is a convex fuzzy open set. □

**Definition 3.20.** *[8] If $\mathcal{X} \neq \emptyset$, then a collection $\mathfrak{J}$ of fuzzy subsets of $\mathcal{X}$ is a fuzzy topology or simply FT on $\mathcal{X}$ if:*
*(1) $\mathcal{X}$ and $\emptyset$ belongs to $\mathfrak{J}$,*
*(2) if $\{\mathcal{B}_i$: i∈$I$ $\}\in \mathfrak{J}$, then $\cup_{i\in I}\mathcal{B}_i \in \mathfrak{J}$,*
*(3) if $\mathcal{B}_1$, $\mathcal{B}_2$,..., $\mathcal{B}_k \in \mathfrak{J}$, then $\cap_{i=1}^{k}\mathcal{B}_i \in \mathfrak{J}$.*

**Theorem 3.21.** *If ($\mathcal{U}$, $\mathfrak{N}$) is a convex fuzzy normed space, then it is a fuzzy topological space.*

*Proof.* If ($\mathcal{U}$, $\mathfrak{N}$) is a convex fuzzy normed space, then putting $\mathfrak{J}_m$= { $\mathcal{W} \subset \mathcal{U}$: $w \in \mathcal{W} \Longleftrightarrow \exists\ \alpha \in (0,1)$ with cfb($w$, $\alpha$) $\subset \mathcal{W}$ }. Therefore, $\mathfrak{J}_m$ must be a fuzzy topology on $\mathcal{U}$.
(1) $\emptyset$, $\mathcal{U} \in \mathfrak{J}_m$;
(2) Suppose that { $\mathcal{B}_i$: i∈I } $\in \mathfrak{J}_m$, then $\cup_{i\in I}\mathcal{B}_i \in \mathfrak{J}_m$ by Definition 3.20.
(3) Let $\mathcal{B}_1$, $\mathcal{B}_2$,..., $\mathcal{B}_k \in \mathfrak{J}_m$then $\cap_{i=1}^{k}\mathcal{B}_i \in \mathfrak{J}_m$ by Definition 3.20.
Hence ($\mathcal{U}$, $\mathfrak{J}_m$) is a fuzzy topological space. □

**Definition 3.22.** *A convex fuzzy normed space ($\mathcal{U}$, $\mathfrak{N}$) is said to be a convex fuzzy complete if for every convex fuzzy Cauchy sequence ($u_k$) in $\mathcal{U}$, $u_k \to u \in \mathcal{U}$ as $k \to infty$.*

**Theorem 3.23.** *In a convex fuzzy normed space ($\mathcal{U}$, $\mathfrak{N}$) if $u_k \to u \in \mathcal{U}$, as $k \to infty$, then ($u_k$) is convex fuzzy Cauchy.*

*Proof.* Suppose that ($u_k$) in $\mathcal{U}$ and $u_k \to u \in \mathcal{U}$ as $k \to infty$. Then, for any $\sigma \in$(0, 1), we can find $N$ with $\mathfrak{N}(u_k -$ u) $< \sigma$ , for all $k \geq N$. Now, for each $m, k \geq N$ $\mathfrak{N}(u_k - u_m) \leq \alpha\mathfrak{N}(u_k - \mathrm{u}) + \beta\mathfrak{N}(\mathrm{u} - u_m)$, where $\alpha + \beta$=1. Therefore, $\mathfrak{N}(u_k - u_m) < \alpha\sigma + \beta\sigma = \sigma$. Hence $(u_k)$ is a convex fuzzy Cauchy sequence. □

**Theorem 3.24.** *In a convex fuzzy normed space ($\mathcal{U}$, $\mathfrak{N}$), if $(u_k) \in \mathcal{U}$ with $u_k \to u$ and $(d_k) \in \mathcal{U}$ with $\mathfrak{N}$ $(u_k - d_k) \to 0$ as $k \to \infty$, then $d_k \to u$.*

*Proof.* Since $u_k \to u$, $\mathfrak{N}(u_k - \mathrm{u}) \to 0$ as $k \to \infty$. Now, $\mathfrak{N}(d_k - u) \leq \alpha\mathfrak{N}(d_k - u_k) + \beta\mathfrak{N}(u_k - \mathrm{u}) \to 0$ as $k \to \infty$, where $\alpha + \beta$=1. Hence $d_k \to u$, as $k \to infty$. □

**Theorem 3.25.** *Let ($\mathcal{U}$, $\mathfrak{N}$) be a convex fuzzy normed space with $\mathcal{D} \subset \mathcal{U}$. Then $d \in \overline{\mathcal{D}}$ if and only if there is $(\mathrm{d_k}) \in \mathcal{D}$ with $\mathrm{d_k} \to d$ as $k \to \infty$.*

*Proof.* Suppose that $d \in \overline{\mathcal{D}}$. If $d \in \mathcal{D}$, then choose the sequence of that type is $(d, d, \ldots, d, \ldots)$. If $d \notin \mathcal{D}$, then construct the sequence $(\mathrm{d_k}) \in \mathcal{D}$ by $\mathfrak{N}(\mathrm{d_k} - d)$¡ $\frac{1}{\mathrm{k}}$, for each $k = 1, 2, 3, \ldots$. Then the convex fuzzy ball cfb$(d, \frac{1}{\mathrm{k}})$ contains $\mathrm{d_k} \in D$ and $\mathrm{d_k} \to d$ because $\lim_{\mathrm{k}\to\infty}\mathfrak{N}(\mathrm{d_k} - d) = 0$.
Conversely, if $(\mathrm{d_k})$ in $\mathcal{D}$ and $\mathrm{d_k} \to d$, then $d \in \mathcal{D}$ or every convex fuzzy ball of $d$ containing points $\mathrm{d_k} \neq \mathrm{d}$, so that $d$ is an accumulation point of of $\mathcal{D}$. Hence $\mathrm{d} \in \overline{\mathcal{D}}$ by the definition of closure. □

**Theorem 3.26.** *Let ($\mathcal{U}$, $\mathfrak{N}$) be a convex fuzzy normed space with $\mathcal{D} \subset \mathcal{U}$. Then $\overline{\mathcal{D}} = \mathcal{U}$ if and only if for any $u \in \mathcal{U}$ there is $d \in D$ with $\mathfrak{N}(u - d) < \alpha$, for some $\alpha \in (0, 1)$.*

*Proof.* Suppose that $\mathcal{D}$ is fuzzy dense in $\mathcal{U}$ and $u \in \mathcal{U}$. So $u \in \overline{\mathcal{D}}$ and by Theorem 3.25 there is a sequence $(\mathrm{d_n}) \in \mathcal{D}$ such that $\mathrm{d_n} \to u$; that is, for any $\alpha \in (0, 1)$, we can find $N$ with $\mathfrak{N}(\mathrm{d_k} - \mathrm{u}) < \alpha$ for all $k \geq N$. Take $d = \mathrm{d_N}$. So $\mathfrak{N}(u - d) < \alpha$.
Conversely, to prove $\mathcal{D}$ is fuzzy dense in $\mathcal{U}$, we have to show that $\mathcal{D} \subseteq \overline{\mathcal{D}}$. Let $u \in \mathcal{U}$. Then there is $\mathrm{d_k} \in \mathcal{D}$ such that $\mathfrak{N}(\mathrm{d_k} - \mathrm{u}) < \frac{1}{\mathrm{k}}$. Now, take $0 < \sigma < 1$ such that $\frac{1}{\mathrm{k}} < \sigma$, for each $k \geq \mathrm{N}$ for $N \in \mathbb{N}$. Hence we have a sequence $(\mathrm{d_k}) \in \mathcal{D}$ such that $\mathfrak{N}(\mathrm{d_k}$ -u$) < \frac{1}{\mathrm{k}} < \sigma$ for all $k \geq N$; that is, $\mathrm{d_k} \to u$ so $u \in \overline{\mathcal{D}}$. □

**Theorem 3.27.** *If ($\mathcal{U}_1$, $\mathfrak{N}_1$) and ($\mathcal{U}_2$, $\mathfrak{N}_2$) are two convex fuzzy normed spaces, then ($\mathcal{U}$, $\mathfrak{N}$) is a convex fuzzy complete convex fuzzy normed space if and only if ($\mathcal{U}_1$,$\mathfrak{N}_1$) and ($\mathcal{U}_2$, $\mathfrak{N}_2$) are fuzzy complete, where $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$ and $\mathfrak{N}[(u_1, u_2)] = \gamma\mathfrak{N}_1(u_1) + \delta\mathfrak{N}_2(u_2)$, for all $(u_1, u_2) \in \mathcal{U}$, where $\gamma + \delta$=1.*

*Proof.* Let ($\mathcal{U}_1$, $\mathfrak{N}_1$) and ($\mathcal{U}_2$, $\mathfrak{N}_2$) be convex fuzzy complete convex fuzzy normed spaces. Let $(u_k)$ be convex fuzzy Cauchy sequence in $\mathcal{U}$ then $(u_k) =$ ( $u_{1k}$, $u_{2k}$), where $(u_{1k}) \in \mathcal{U}_1$ and $(u_{2k}) \in \mathcal{U}_2$. Hence $\mathfrak{N}(u_k - u_m)$ convex fuzzy converges to zero as $k \to \infty$ and $m \to \infty$ this implies that $[\gamma\mathfrak{N}_1(u_{1k} - u_{1m}) + \delta\mathfrak{N}_2(u_{2k} - u_{2m})]$ is convex fuzzy convergent to zero as $k \to \infty$ and $m \to \infty$. Hence $\mathfrak{N}_1(u_{1k} - u_{1m})$ is convex fuzzy convergent to zero in ($\mathcal{U}_1$, $\mathfrak{N}_1$) as $k \to \infty$ and $m \to \infty$ and $\mathfrak{N}_2(u_{2k} - u_{2m})$ is convex fuzzy convergent to zero in ($\mathcal{U}_2$, $\mathfrak{N}_2$) as $k \to \infty$ and $m \to \infty$. Therefore, $(u_{1k})$ is a fuzzy Cauchy sequence in ($\mathcal{U}_1$, $\mathfrak{N}_1$) and $(u_{2k})$ is a fuzzy Cauchy sequence in ($\mathcal{U}_2$, $\mathfrak{N}_2$). But ($\mathcal{U}_1$, $\mathfrak{N}_1$) and ($\mathcal{U}_2$, $\mathfrak{N}_2$,) are convex fuzzy complete. So there is $u_1 \in \mathcal{U}_1$ and $u_2 \in \mathcal{U}_2$ such that $(u_{1k})$ convex fuzzy converges to $u_1 \in \mathcal{U}_1$ and $(u_{2k})$ convex fuzzy converges to $u_2 \in \mathcal{U}_2$. Put $u = (u_1, u_2)$. Then $u \in \mathcal{U}$ and $(u_k)$ is convex fuzzy convergent to $u \in \mathcal{U}$ since $\mathfrak{N}(u_k$ −u)= $\mathfrak{N}[(u_{1k}, u_{2k}) - (u_1, u_2)] = \mathfrak{N}$ $[(u_{1k} - u_1) + (u_{2k} - u_2)] = \gamma\mathfrak{N}_1[(u_{1k} - u_1)] + \delta\mathfrak{N}_2[(u_{2k} - u_2)]$ By taking the limit on both sides as $k \to \infty$, we have $\mathfrak{N}(u_k$ −u) →0.
The proof of the converse is similar and hence is omitted. □

The following two corollaries follow easily.

**Corollary 3.28.** *If ($\mathcal{U}_1$, $\mathfrak{N}_1$), ($\mathcal{U}_2$, $\mathfrak{N}_2$), ..., ($\mathcal{U}_k$, $\mathfrak{N}_k$) are convex fuzzy normed spaces, then ($\mathcal{U}$, $\mathfrak{N}$) is convex fuzzy complete convex fuzzy normed space if and only if ($\mathcal{U}_1$, $\mathfrak{N}_1$), ($\mathcal{U}_2$, $\mathfrak{N}_2$), ..., ($\mathcal{U}_k$, $\mathfrak{N}_k$) are convex fuzzy complete, where* $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_k$ *and* $\mathfrak{N}[(u_1, u_2, \ldots, u_k)] = \delta_1\mathfrak{N}_1(u_1) + \delta_2 \mathfrak{N}_2(u_2) + \ldots + \delta_k\mathfrak{N}_k(u_k)$ *for all* $(u_1, u_2, \ldots, u_k) \in \mathcal{U}$, *where* $\delta_1 + \delta_2 + \ldots + \delta_k = 1$.

**Corollary 3.29.** *Let ($\mathcal{U}$, $\mathfrak{N}$) be a convex fuzzy normed space. Then ($\mathcal{U}^k$, $\mathfrak{N}_k$) is a convex fuzzy complete convex fuzzy normed space if and only if ($\mathcal{U}$, $\mathfrak{N}$) is convex fuzzy complete, where* $\mathcal{U}^k = \mathcal{U} \times \mathcal{U} \times \ldots \times \mathcal{U}$ *[k-times], where* $k \in \mathbb{N}$ *and* $\mathfrak{N}_k[(u_1, u_2, \ldots, u_k)] = \delta_1 \mathfrak{N}(u_1) + \delta_2\mathfrak{N}(u_2) + \ldots + \delta_k\mathfrak{N}(u_k)$, *for all* $(u_1, u_2, \ldots, u_k) \in \mathcal{U}^k$, *where* $\delta_1 + \delta_2 + \ldots + \delta_k = 1$.

# 4 Conclusion

In this paper, we opened a new line of research in fuzzy functional analysis by introducing a new fuzzy metric space, called a convex fuzzy metric space. Moreover, we introduced the notion of convex fuzzy absolute value which is a generalization of the ordinary absolute value.

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