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New Fundamental Inequalities for Fractional Ambiguity Function

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Abstract

In this present work, we establish new fundamental uncertainty principles for the fractional ambiguity function. These principles are generalized forms of inequalities concerning the Fourier transform.

1 Introduction

The fractional Fourier transform is a powerful mathematical tool that extends the classical Fourier transform. It has been used in various fields including signal processing, optics, and quantum mechanics. In [1, 2, 4], the authors proposed the linear canonical ambiguity function, which is an extension of the ambiguity function using the linear canonical transform. Inspired

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by the idea, the authors [3] have generalized the ambiguity function to the fractional Fourier transform called the fractional ambiguity function. They investigated its properties and applied them in signal processing. However, they did not publish yet the uncertainty principle related to the transformation. Therefore, the main objective of this paper is to propose the uncertainty principles concerning the fractional ambiguity function (FrAF).

2 Main Result

In following, we introduce a definition of the fractional ambiguity function (FrAF).

Definition 2.1. *Let $f, g \in L^2(\mathbb{R})$. The fractional ambiguity function (FrAF) is defined as*

$$\mathcal{A}^{\theta}_{f,g}(x,\omega) = \int_{\mathbb{R}} f\left(t+\frac{x}{2}\right) g\left(t-\frac{x}{2}\right) K^{\theta}(t,\omega)\, dt, \tag{2.1}$$

where the kernel $K^{\theta}(t,\omega)$ is given by

$$K^{\theta}(t,\omega) = C_{\theta} e^{i\left(t^2+\omega^2\right)\frac{cot\theta}{2} - it\omega csc\theta}, \quad C_{\theta} = \sqrt{\frac{1-i\cot\theta}{2\pi}},\ \theta \neq n\pi. \tag{2.2}$$

It is easy to verify that the relation between the fractional ambiguity function and the Fourier transform is

$$\left(C^{\theta}\right)^{-1} \mathcal{A}^{\theta}_{f,g}(x,\omega)\, e^{-i\omega^2\frac{\cot\theta}{2}} = \mathcal{F}\left\{h^{\theta}_{f,g}(x,t)\right\}(\omega\csc\theta), \tag{2.3}$$

where $h^{\theta}_{f,g}(x,t) = f\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)} e^{i\frac{\cot\theta}{2}t^2}$ and $\mathcal{F}\{f\}(\omega) = \int_{\mathbb{R}} f(t)\, e^{-i\omega t} dt$.

Theorem 2.2. *Let f, g be two functions belonging to $L^2(\mathbb{R})$ such that $\|f\|_{L^2(\mathbb{R})} = \|g\|_{L^2(\mathbb{R})} = 1$. Let $T \subseteq \mathbb{R}\times\mathbb{R}$ be a measurable subset. If*

$$\int_T \int_T \left|\mathcal{A}^{\theta}_{f,g}(x,\omega)\right| dx d\omega \geq 1-\xi, \tag{2.4}$$

then, for every $\xi \geq 0$,

$$1-\xi \leq \frac{1}{\sqrt{2\pi}|\sin\theta|}\mu(T), \tag{2.5}$$

where $\mu(T)$ is the Lebesgue measure of T.

Proof. Using Hölder inequality, we get

$$\begin{aligned}\left|\mathcal{A}^{\theta}_{f,g}\left(x,\omega\right)\right| &= \left|\int_{\mathbb{R}} C_\theta f\left(t+\frac{x}{2}\right)\overline{g\left(t-\frac{x}{2}\right)}e^{i\left(t^2+\omega^2\right)\frac{cot\theta}{2}-it\omega csc\theta}dt\right| \\ &\leq \frac{1}{\sqrt{2\pi}\left|\sin\theta\right|}\left\|f\right\|_{L^p(\mathbb{R})}\left\|g\right\|_{L^q(\mathbb{R})}, \quad \frac{1}{p}+\frac{1}{q}=1. \end{aligned} \tag{2.6}$$

For $p = 2$ and $q = 2$, we have

$$\left|\mathcal{A}^{\theta}_{f,g}\left(x,\omega\right)\right| \leq \frac{1}{\sqrt{2\pi}|\sin\theta|}\left\|f\right\|_{L^2(\mathbb{R})}\left\|g\right\|_{L^2(\mathbb{R})}. \tag{2.7}$$

This implies that

$$\begin{aligned} 1-\xi &\leq \int_T\int_T \left|\mathcal{A}^{\theta}_{f,g}\left(x,\omega\right)\right| dxd\omega \leq \frac{1}{\sqrt{2\pi}|\sin\theta|}\left\|f\right\|_{L^2(\mathbb{R})}\left\|g\right\|_{L^2(\mathbb{R})} \\ &= \frac{1}{\sqrt{2\pi}|\sin\theta|}\mu(T), \end{aligned}$$

where $\|f\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}}|f(x)|^2dx\right)^{\frac{1}{2}}$. Thus the proof is complete. ☐

Theorem 2.3. *Let f, g be two functions and $f \in L^2(\mathbb{R})$. Then, for all $r \in [1,\infty)$,*

$$\begin{aligned} &\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{A}^{\theta}_{f_1,g_1}\left(x,\omega\right)\mathcal{A}^{\theta}_{f_2,g_2}\left(x,\omega\right)\right|^r d\omega dx\right)^{\frac{1}{r}} \\ &\qquad \leq \left(\frac{1}{\sqrt{2\pi}|\sin\theta|}\right)^{\frac{r-1}{r}}\left\|f_1\right\|_{L^2(\mathbb{R})}\left\|g_1\right\|_{L^2(\mathbb{R})}\left\|f_2\right\|_{L^2(\mathbb{R})}\left\|g_2\right\|_{L^2(\mathbb{R})}. \end{aligned} \tag{2.8}$$

Proof. According to the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{A}^{\theta}_{f_1,g_1}\left(x,\omega\right)\mathcal{A}^{\theta}_{f_2,g_2}\left(x,\omega\right)\right| d\omega dx \\ &\leq \left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{A}^{\theta}_{f_1,g_1}\left(x,\omega\right)\right|^2 d\omega dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{A}^{\theta}_{f_2,g_2}\left(x,\omega\right)\right|^2 d\omega dx\right)^{\frac{1}{2}}. \end{aligned} \tag{2.9}$$

It is not difficult to check that

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{A}^{\theta}_{f_1,g_1}\left(x,\omega\right)\mathcal{A}^{\theta}_{f_2,g_2}\left(x,\omega\right)\right| d\omega dx \leq \left\|f_1\right\|_{L^2(\mathbb{R})}\left\|g_1\right\|_{L^2(\mathbb{R})}\left\|f_2\right\|_{L^2(\mathbb{R})}\left\|g_2\right\|_{L^2(\mathbb{R})}.$$

Thus,

$$\begin{aligned}
&\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{A}^{\theta}_{f_1,g_1}(x,\omega)\,\mathcal{A}^{\theta}_{f_2,g_2}(x,\omega)\right|^{r}d\omega dx\right)^{\frac{1}{r}} \\
&\qquad \le \left\|\mathcal{A}^{\theta}_{f_1,g_1}\mathcal{A}^{\theta}_{f_2,g_2}\right\|^{\frac{r-1}{r}}_{L^{\infty}(\mathbb{R}\times\mathbb{R})}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{A}^{\theta}_{f_1,g_1}(x,\omega)\,\mathcal{A}^{\theta}_{f_2,g_2}(x,\omega)\right|d\omega dx\right)^{\frac{1}{r}} \\
&\qquad \le \left(\frac{1}{\sqrt{2\pi}|\sin\theta|}\|f_1\|_{L^2(\mathbb{R})}\|g_1\|_{L^2(\mathbb{R})}\|f_2\|_{L^2(\mathbb{R})}\|g_2\|_{L^2(\mathbb{R})}\right)^{\frac{r-1}{r}} \\
&\qquad\quad \left(\|f_1\|_{L^2(\mathbb{R})}\|g_1\|_{L^2(\mathbb{R})}\|f_2\|_{L^2(\mathbb{R})}\|g_2\|_{L^2(\mathbb{R})}\right)^{\frac{1}{r}}
\end{aligned}$$

which completes the proof. □

Corollary 2.4. *Let f, g be two functions. Then, for every $f \in L^2(\mathbb{R})$ with $r \in [2,\infty)$,*

$$\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{A}^{\theta}_{f,g}(x,\omega)\right|^{s}dxd\omega\right)^{\frac{1}{s}} \le \left(\frac{1}{\sqrt{2\pi}|\sin\theta|}\right)^{\frac{\frac{s}{2}-1}{s}}\|f\|^2_{L^2(\mathbb{R})}\|g\|^2_{L^2(\mathbb{R})}. \tag{2.10}$$

Proof. For $r=\infty$, we have

$$\left|\mathcal{A}^{\theta}_{f,g}(x,\omega)\right| \le \frac{1}{\sqrt{2\pi}|\sin\theta|}\|f\|_{L^2(\mathbb{R}^2)}\|g\|_{L^2(\mathbb{R}^2)}. \tag{2.11}$$

For all $r\in[1,\infty)$,

$$\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{A}^{\theta}_{f,g}(x,\omega)\right|^{2r}dxd\omega\right)^{\frac{1}{r}} \le \left(\frac{1}{\sqrt{2\pi}|\sin\theta|}\right)^{\frac{r-1}{r}}\|f\|^2_{L^2(\mathbb{R})}\|g\|^2_{L^2(\mathbb{R})}. \tag{2.12}$$

Now putting $s=2r\in[2,\infty)$ yields

$$\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{A}^{\theta}_{f,g}(x,\omega)\right|^{s}dxd\omega\right)^{\frac{1}{s}} \le \left(\frac{1}{\sqrt{2\pi}|\sin\theta|}\right)^{\frac{\frac{s}{2}-1}{s}}\|f\|^2_{L^2(\mathbb{R})}\|g\|^2_{L^2(\mathbb{R})}.$$

This finishes the proof. □

Below we obtain a generalization of Theorem 2.2 mentioned above.

Theorem 2.5. *With the notations of Theorem 2.3, for all* $r > 2$*, the following inequality holds*

$$\mu(T) \geq (1-\xi)^{\frac{r}{r-2}}(\sqrt{2\pi}|\sin\theta|). \tag{2.13}$$

Proof. Applying Hölder inequality, we obtain

$$\begin{aligned} 1-\xi &\leq \int_T \int_T \left|\mathcal{A}^{\theta}_{f,g}(x,\omega)\right|^2 dx d\omega \\ &\leq \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\mathcal{A}^{\theta}_{f,g}(x,\omega)\right|^{2\frac{r}{2}} dx d\omega\right)^{\frac{2}{r}} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \left(\chi_T(x,\omega)\right)^{\frac{r}{r-2}} d\omega dx\right)^{\frac{r-2}{r}}, \end{aligned} \tag{2.14}$$

where χ_T is the function of T. Substituting relation (2.10) into the right-hand sides of equation (2.14) yields

$$1-\xi \leq \left(\left(\frac{1}{\sqrt{2\pi}|\sin\theta|}\right)^{\frac{\frac{r}{2}-1}{r}} \|f\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}\right)^2 (\mu(T))^{\frac{r-2}{r}}. \tag{2.15}$$

For all $r > 2$, we get

$$1-\xi \leq \left(\frac{1}{\sqrt{2\pi}|\sin\theta|}\right)^{\frac{r-2}{r}} (\mu(T))^{\frac{r-2}{r}}. \tag{2.16}$$

Hence

$$(1-\xi)^{\frac{r}{r-2}}(\sqrt{2\pi}|\sin\theta|) \leq \mu(T).$$

This proves the theorem. ∎

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