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On Some Graphs Based on the Ideals of KU-algebras

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Abstract

In this paper, we will construct a few types of graphs based on the ideal-annihilator, right-ideal-annihilator, left-ideal-annihilator for a KU-algebras. We will also study some graph invariants, such as connectivity, regularity, and planarity for these graphs.

1 Introduction

The motivation of logical algebras arises from the work on BCI/BCK algebras. These algebras were introduced by Imai and Iseki [4] as a generalization of the set-theoretic difference and proportional calculi.
Algebraic combinatorics is an area of mathematics that employs methods of abstract algebra in various combinatorial contexts and vice versa. Associating a graph to an algebraic structure is a research subject in this area and it has attracted considerable attention. The research in this subject aims at exposing the relationship between algebra and graph theory and at advancing the application of one to the other. The story goes back to a paper by

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Beck [1] in 1998, where he introduced the idea of a zero-divisor graph of a commutative ring R with identity. He defined $\Gamma(R)$ to be the graph whose vertices are elements of R and in which two vertices are adjacent if and only if $xy = 0$ Recently, Halas and Jukl [3] introduced the zero divisor graphs of posets. In this paper, we deal with zero-divisor graphs of KU-algebras based on an ideal. In 1966, Imai and Iseki [4] introduced the notion of BCK-algebra. In the same year, Iseki [5] introduced BCI-algebra as a superclass of the class of BCK- algebras. Jun and Lee [6] defined the notion of associated graph of BCK- algebra and verified some properties of this graph. Tahmasbpour [9] studied chordality of the graph defined by Zahiri and Borzooei and introduced four types of graphs of BCK-algebras which are constructed by equivalence classes determined by an ideal I. Moreover, Tahmasbpour [9] introduced some new graphs of BCK-algebras based on a fuzzy ideal μ_1. Futhermore, Tahmasbpour [10] introduced twelve kinds of graphs of lattice implication algebras based on filter and LI-ideal.

A class of logical algebras, namely KU-algebras, was introduced by Prabpayak and Leerawat [11]. Some basic algebraic properties like homomorphisms of KU-algebras and related structural properties were presented in [11] and [12]. Later, the KU-algebras were studied by several authors and contributed to the study through different means such as fuzzy, neutrosophic and intuitionistic context, soft and rough sense, etc. Naveed et al. [13] have introduced the notion of cubic KU-ideals of KU-algebras whereas Mostafa et al. [14] defined fuzzy ideals of KU-algebras. Moreover, Mostafa et al. [15] studied Interval-valued fuzzy KU-ideals in KU-algebras. Moin and Ali introduced roughness in KU-algebras [16]. Ali et al. [17] constructed a pseudo-metric on KU-algebras and studied its properties. Senapati and Shum [18] defined Atanassov's intuitionistic fuzzy bi-normed KU-ideals of a KU-algebra.

The next section is an introduction to a general theory of KU-algebras. We will first discuss the notions of KU-algebras and then investigate their elementary and fundamental properties. Moreover, we will consider basic notions, such as ideals and ideal annihilators among others. In the remaining sections, we will study the graphs of KU-algebras which are constructed from ideal-annihilator, denoted by $\phi_1(X)$. We will also explore the graphs of KU-algebras that are constructed from right ideal-annihilator, left ideal-annihilator, denoted by $\Delta_1(X)$ and $\Sigma_1(X)$. In the last section, we will introduce the associated graph $Y_1(X)$ that is constructed from some binary operations on the elements of KU-algebras.

2 Preliminaries of KU-algebras and Graph Theory

In this section, some basic definitions, notations and properties related to KU-algebras are given.

Definition 2.1. [11] An algebra $(X, \centerdot, 1)$ of type $(2, 0)$ with a binary operation $\centerdot$ satisfying the following identities for any $u, v, w \in X$, is called a KU-algebra.

(ku1) $(u \centerdot v) \centerdot [(v \centerdot w) \centerdot (u \centerdot w)] = 1$,
(ku2) $u \centerdot 1 = 1$,
(ku3) $1 \centerdot u = u$,
(ku4) $u \centerdot v = v \centerdot u = 1$ implies $u = v$.

We denote a KU-algebra by $(X, \centerdot, 1)$ and otherwise will be specified. For simplicity, we will call X a KU-algebra. A fixed element 1 of X is called the constant element. A partial order " $\leq$ " on X is defined as $u \leq v$ if and only if $v \centerdot u = 1$.

Lemma 2.2. [11] $(X, \centerdot, 1)$ is a KU-algebra if and only if it satisfies following conditions:

(ku5) $(v \centerdot w) \centerdot (u \centerdot w) \leq u \centerdot v$,
(ku6) $1 \leq u$,
(ku7) $u \leq v$, $v \leq u$ implies $u = v$,

Lemma 2.3. [14] The following identities hold in any KU-algebra:

(1) $w \centerdot w = 1$,
(2) $w \centerdot (u \centerdot w) = 1$,
(3) $u \leq v$ implies $v \centerdot w \leq u \centerdot w$,
(4) $w \centerdot (v \centerdot u) = v \centerdot (w \centerdot u)$, for all $u, v, w \in X$,
(5) $v \centerdot [(v \centerdot u) \centerdot u] = 1$.

Example 1. [14] Consider a set $X = \{1, x, y, z, w\}$ with the binary operation $\centerdot$ defined by the given table

$\centerdot$	1	x	y	z	w
1	1	x	y	z	w
x	1	1	y	z	w
y	1	x	1	z	z
z	1	1	y	1	y
w	1	1	1	1	1

It can be easily verified that X with a binary operation $\centerdot$ forms a KU-algebra.

Definition 2.4. [11] A KU-ideal I of a KU-algebra X is a non-empty subset of X satisfying the following conditions:

(1) $1 \in I$,

(2) $u \centerdot v \in I$, and $u \in I \Rightarrow v \in I$, for all $u, v \in X$.

Definition 2.5. *An ideal P of X is called prime if $(v \centerdot u) \centerdot u \in P$ implies $u \in P$ or $v \in P$.*

Definition 2.6. *A KU-algebra X is said to be a bounded KU-algebra if there exists an element $t \in X$ such that $u \leq t$ for all $u \in X$. The element t is said to be the unit element of X. In a bounded KU-algebra, we use the notations $Nu = u \centerdot t$ and the set $N(X) = \{Nu | u \in X\}$.*

Theorem 2.7. *Let X be a bounded KU-algebra with a unit element t. Then the following statements hold for any $u, v \in X$:*

1. *$N1 = t$ and $Nt = 1$.*

2. *$Nv \centerdot Nu \leq u \centerdot v$.*

3. *$v \leq u$ implies $Nu \leq Nv$.*

Proof. 1. By using (ku3) and Lemma 2.3(1), $N1 = 1 \centerdot t = t$ and $Nt = t \centerdot t = 1$.
2. $Nv \centerdot Nu = (v \centerdot t) \centerdot (u \centerdot t) \leq u \centerdot v$ by (ku5).
3. If $v \leq u$, then, by Lemma 2.3(3), we get $u \centerdot t \leq v \centerdot t = Nu \leq Nv$. □

Definition 2.8. *In a KU-algebra X, we define $u \wedge v = (v \centerdot u) \centerdot u$, and $u \vee v = N(Nu \wedge Nv)$.*

Definition 2.9. [2] $G = (V(G), E(G))$ is called a graph where $V(G)$ is called the set of vertices and $E(G)$ is called the set of edges of G.

In this article, we will consider only simple graphs. A simple graph is a graph with no multiple edges and loops.

Definition 2.10. *A graph H is called a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.*

Definition 2.11. *A complete graph G is graph in which two distinct vertices are adjacent by exactly one edge. The greatest induced complete subgraph in a graph G is called a clique of G. If a graph G has a clique with n elements, then we say that the graph G has clique number n and we write $\omega(G) = n$.*

Definition 2.12. *If there is a path connecting any two vertices in a graph G, then it is said to be connected; otherwise, it is said to be not connected. Let $d(u, v)$ be the length of the shortest path between u to v for any distinct vertices u and v of G. If there is no such path, then we define $d(x, y) \to \infty$.*

Definition 2.13. *The neighborhood of a vertex $v \in V(G)$ is the set $N_G(v) = \{x \in V(G) | \ xv \in E(G)\}$. Moreover, $|N_G(v)|$ is called the degree of vertex v. A graph G is said to be a regular of degree d if every vertex has degree d. Furthermore, for distinct vertices u and v, the notation $u - v$ is used to show that there is a path between u and v.*

Definition 2.14. [1] We say that the chromatic number of G is K and write $\chi(G) = K$ if K is the minimum number of colors needed to color the vertices of G so that no two adjacent vertices share the same color. Note that $\chi^{(G)} \geq \omega(G)$.

Definition 2.15. [2] An Euler path in G is a closed walk in a graph G that contains all of G's edges. An Euler graph is a graph that contains an Euler line. An Euler graph is a connected graph because the Euler line (which is a walk) comprises all of the graph's edges. The connected graph G is Eulerian if and only if all of its vertices are of even degree, according to Euler's theorem.

Definition 2.16. [2] A subdivision of a graph is a graph that is obtained by replacing edges with pathways from the original graph. If a graph G can be drawn in a plane without the edges crossing, then it is said to be planar.

Redrawing the edges in such a way that no edges cross is equivalent to proving that a graph is planar. The vertices may need to be moved around, and the edges will have to be drawn in a very indirect manner. A finite graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$, according to Kuratowski's theorem. Any planar graph has a clique number of less than or equal to four.

Definition 2.17. [8] Assume that G is a planar graph. A face of a graph is a region that is bounded by its edges. If an undirected graph can be drawn in the plane without crossing in such a way that all of the vertices belong to the drawing's unbounded face, then it is called an outer planar graph. A graph is said to be outer planar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.

Definition 2.18. [7] If a surface is homeomorphic to a sphere with g handles or equivalently holes, then the number g is called the genus of the surface.

In addition, the genus g of a graph G is the smallest genus of all surfaces, allowing the graph G to be drawn on it without any edge crossing. Because the genus of a plane is zero, the graphs of genus zero are exactly planar graphs. The graphs that can be drawn on a torus without edge-crossing are called toroidal. They have a genus of one since the genus of a torus is one. The genus of a graph G is denoted by $y(G)$.

Theorem 2.19. [2] Let m and n be two positive integers. We have:

1. $\gamma(K_n) = \lceil (1/12)(n-3)(n-4) \rceil$ if $n \geq 3$,
2. $\gamma(K_{m,n}) = \lceil (1/4)(m-2)(n-2) \rceil$ if $m, n \geq 2$

3 Graphs Based on the Ideal-annihilator

In this section, we will define a few graphs on the ideals of a KU-algebra and will discuss properties of those graphs.

Definition 3.1. *For a nonempty subset A of a KU-algebra X and an ideal I of X, the set of all zero-divisors of A by I is defined as:*

$$Ann_I(A) = \{u \in X |\ a \centerdot u \in I \text{ or } u \centerdot a \in I, \forall a \in A\}.$$

Proposition 3.2. *For any two nonempty subsets A and B of a KU-algebra X and an ideal I of X, the following hold:*

1. $\{1\} \subseteq Ann_I(A)$.
2. $I \subseteq Ann_I(A)$.
3. *If* $A \subseteq B$, *then* $Ann_I B \subseteq Ann_I(A)$.
4. *If* $1 \in A$, *then* $Ann_I(A) = Ann_1(A - \{1\})$.
5. $Ann_I(I) = X$.
6. *If* $I = \{1\}$, *then* $Ann_I(A) = \{x |\ x$ *is comparable to every element in* $A\}$.

Proof. 1. By (ku2) and Definition 2.4 (1), $a \centerdot 1 = 1 \in I$ for all $a \in A$ and hence $\{1\} \subseteq Ann_I(A)$.

2. Let $u \in I$. Then, by Definition 2.1, we have $a \centerdot u \in I, \forall a \in A$. Also, $1 \centerdot u = 0, \forall u \in X$, So $I \cup \{1\} \subseteq Ann_I(A)$.

3. Suppose that $u \in Ann_I B$. Then $b \centerdot u \in I$ or $u \centerdot b \in I$, $\forall b \in B$. But $A \subseteq B$. Therefore, $b \centerdot u \in I$ or $u \centerdot b \in I, \forall b \in A$; i.e., $u \in Ann_I(A)$. Hence $Ann_I B \subseteq Ann_I(A)$.

4. According to Definition 2.1 we have $Ann_I(A) = \cap_{a \in A} Ann_I a$. Also, $Ann_I\{1\} = X$. Then $Ann_I(A) = Ann_I(A - \{1\})$.

5. Let $u \in X$. By Definition 2.1, $u \centerdot a \in I, \forall a \in I$. Then $u \in Ann_I(I)$ and so $Ann_I(I) = X$.

6. Follows from the definitions.

□

Definition 3.3. *Let I be an ideal of X. Then $\phi_I(X)$ is a simple graph, with vertex set X and two distinct vertices x and y being adjacent if and only if $Ann_I\{x, y\} = I \cup \{1\}$.*

Theorem 3.4. *Let I be an ideal of X, then $N_G(\{1\}) = \phi$, where $G = \phi_I(X)$.*

Proof. We know that $Ann_I\{1\} = X$ and for all $x \in X, x \neq 1$, we have $I \cup \{x, 1\} \subseteq Ann_I\{x\}$. Then $I \cup \{x, 1\} \subseteq Ann_I\{x\}$ and $I \cup \{x, 1\} \subseteq Ann_I\{x, 1\}$, for all $x \in X, x \neq 1$. So, by Definition 2.1 of graph $\phi_I(X)$, for all $x \in X, x \neq 1, x$ is connected to element 1 if and only if $x \in I$, if $x \in I$. By Proposition 3.2, $Ann_I\{x\} = X$. So the element 1 is not connected to x, for all $x \in X$. □

Theorem 3.5. *Let $X = \{1\} \cup Atom(X)$. $I = \{1\}$ is an ideal of X.*

Proof. We know $Ann_{\{1\}}\{1\} = X$ by Proposition 3.2 since $X = Atom(X) \cup \{1\}$, we have, for all $X \in Atom(X)$, $Ann_{\{1\}}\{x\} = \{1, x\}$. On the other hand we know $Ann_{\{1\}}\{x, y\} = Ann_{\{1\}}\{x\} \cap Ann_{\{1\}}\{y\}$. Then by Definition 3.3 of graph $\phi_{\{1\}}(X)$, x and y are adjacent if and only if $x, y \in Atom(X)$. □

Theorem 3.6. *Let $X = \{1\} \cup Atom(X)$. Then*
$\omega(\phi_{\{1\}}(X)) = |Atom(X)|$.

Proof. It follows from Theorem 3.5. □

Theorem 3.7. *Let $I = \{1\}$ be an ideal of X. Then*
$N_G(x) = \{y; y$ *is not comparable with* $x\}$, *where* $G = \phi_I(X), x \neq 1$.

Proof. For all $x \in X, x \neq 1$, we have
$Ann_{\{1\}}\{x\} = \{y; y$ *is not comparable with* $x\}$.
On the other hand, $Ann_{\{1\}}\{x, y\} = Ann_{\{1\}}\{x\} \cap Ann_{\{1\}}\{y\}$. Then by Definition 3.3 of graph $\phi_{\{1\}}(X)$, x and y are adjacent if and only if x and y are not comparable with each other. □

Theorem 3.8. *Let I be an ideal of X. Then $\alpha(\phi_I(X)) \geq |I|$.*

Proof. We suppose that $x, y \in I$. By Proposition 3.2(5), $Ann_I\{x\} = X$ and $Ann_I\{y\} = X$. Therefore, by Definition 3.3 of graph $\phi_I(X)$, $y \centerdot x \notin E(\phi_I(x))$. Consequently, $\alpha(\phi_I(X)) \geq |I|$. □

Theorem 3.9. *Let $|X| > 2$ and I be a prime ideal of X. Then $\phi_I(X)$ is an empty graph.*

Proof. Suppose, on the contrary, that $\phi_I(X)$ is not an empty graph. Then there exist $x, y \in X$, such that $xy \in E(\phi_I(X))$. So, by Definition 3.3 of graph $\phi_I(X)$, we have $Ann_I\{x, y\} = I \cup \{1\}$. On the other hand, since $|X - I| > 1$, we can choose $z \in X, z \notin 1, z \neq 1$. Since I is a prime ideal, $x \centerdot z \in I$ or $z \centerdot x \in I$, and $y \centerdot z \in I$ or $z \centerdot y \in I$. Hence $z \in Ann_I\{x, y\}$, which is a contradiction. □

4 Graphs of KU-algebras Based on Left and Right Ideal-annihilator

Definition 4.1. *Let I be an ideal of X. Then the sets $Ann_I^R\{x\} = \{y \in X; x \centerdot y \in I\}$, $Ann_I^L\{x\} = \{y \in X; y \centerdot x \in I\}$ are called right-ideal-annihilator and left-ideal-annihilator of x, respectively.*

Definition 4.2. *Let I be an ideal of X. Then $\Sigma_I(X)$ and $\Delta_I(X)$ are two simple graph with vertex set X and two distinct vertices x and y being adjacent in $\Sigma_I(X)$ if and only if $Ann_I^R\{x\} \subseteq Ann_I^R\{y\}$ or $Ann_I^R\{y\} \subseteq Ann_I^R\{x\}$. Also, there is an edge between x and y in the graph $\Delta_I(X)$ if and only if $Ann_I^L\{x\} \subseteq Ann_I^L\{y\}$ or $Ann_I^L\{y\} \subseteq Ann_I^L\{x\}$.*

Example 2. *Let $X = \{1, a, b, c, d\}$ and the operation $\centerdot$ is given by the following table:*

$\centerdot$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	c	c
b	1	1	1	c	c
c	1	1	a	1	a
d	1	1	1	1	1

It is clear that $(X, \centerdot, 1)$ is a bounded KU-algebra of X.
We can see the graphs $\Sigma_{\{1\}}(X)$ and $\Delta_{\{1\}}(X)$ will represent the same graph, $K_5 \setminus \{bc\}$.

Proposition 4.3. *Let I be an ideal of X. Then the following statements hold:*

1. $\omega(\Sigma_I(X)) \geq max\{|A|; A\, is\, a\, chain\, in\, X\}$
2. $\omega(\Delta_I(X)) \geq max\{|A|; A\, is\, a\, chain\, in\, X\}$

Proof. 1. According to Definition 2.1 if $x \leq y$, then $z \centerdot x \leq z \centerdot y$. On the other hand, we let $x \leq y, z \in Ann_I^R\{y\}$. Then, by Definition 4.1, $z \centerdot y \in I$. Therefore, by Definition 2.4, $z \centerdot x \in I$. Hence $z \in Ann_I^R\{x\}$ and so $Ann_I^R\{y\} \subseteq Ann_I^R\{x\}, x \centerdot y \in E(\Sigma_I(X))$.

2. Similar to part (1).

□

Theorem 4.4. *Let I be an ideal X. Then the following statements hold:*

1. *$\Sigma_I(X)$ is connected, $diam(\Sigma_I(X)) \leq 2$, $gr(\Sigma_I(X)) = 3$.*
2. *$\Delta_I(X)$ is connected, $diam(\Delta_I(X)) \leq 2$, $gr(\Delta_I(X)) = 3$.*

Proof. 1. For all $x \in X, x \leq 1$. By Proposition 4.3, the element 1 is connected to every element in X. Therefore, $\Sigma_I(X)$ is connected and hence $diam(\Sigma_I(X)) \leq 2$. Finally, $gr(\Sigma_I(X)) = 3$.

2. Similar to part (1).

□

Theorem 4.5. *Let I be an ideal of X. Then the following statements hold:*

1. *$\Sigma_I(X)$ is regular if and only if it is complete.*
2. *$\Delta_I(X)$ is regular if and only if it is complete.*

Proof. 1. Suppose that $\Sigma_I(X)$ is regular. Since $deg(1) = |X| - 1$, for all $x \in X, deg(x) = |X| - 1$. Hence $\Sigma_I(X)$ is a complete graph. Conversely, a complete graph is always regular.

2. Similar to part (1).

□

Proposition 4.6. *Let X be a chain and let I be an ideal of X. Then the graphs $\Sigma_I(X)$ and $\Delta_I(X)$ are planar if and only if $|X| \leq 4$.*

Proof. Using Proposition 4.3, the graphs $\Sigma_I(X)$ and $\Delta_I(X)$ are complete graphs of K_2 and K_3 for $|x| = 3$ and $|X| = 4$, respectively and hence they are planar for $|X| \leq 4$. Now if $|X| \geq 5$, then $\Sigma_I(X)$ and $\Delta_I(X)$ have a subgraph isomorphic to K_5. Consequently, by Kuratowski's Theorem, the graphs $\Sigma_I(X)$ and $\Sigma_I(X)$ are not planar. Conversely, we know that K_5 has five vertices. Hence if any graph $\Sigma_I(X)$ or $\Delta_I(X)$ is not planar, the graphs $\Sigma_I(X)$ and $\Sigma_I(X), (X)$, have at least five vertices, which is contrary to $|X| \leq 4$. □

Proposition 4.7. *Let X be a chain and let I be an ideal of X. Then the graphs $\Sigma_I(X)$ and $\Delta_I(X)$ are outer planar graphs if and only if $|X| \leq 3$.*

Proof. According to Proposition 4.3, the graphs $\Sigma_I(X)$ and $\Delta_I(X)$ are complete graphs. Now, if $|X| \geq 4$, then both graphs $\Sigma_I(X)$ and $\Delta_I(X)$ have a subgraph that is isomorphic to K_4. By Definition 2.17, the graphs $\Sigma_I(X)$ and $\Delta_I(X)$ are not outer planar. Since K_4 has four vertices, if any of the graphs $\Sigma_I(X)$ or $\Delta_I(X)$ is not outer planar, then the graphs $\Sigma_I(X)$ and $\Delta_I(X)$, have at least four vertices, respectively, which is contrary to the fact that $|X| \leq 3$. □

Proposition 4.8. *Let X be a chain and let I be an ideal of X. Then the graphs $\Sigma_I(X)$ and $\Delta_I(X)$ are toroidal graphs if and only if $|X| \leq 7$.*

Proof. By using Proposition 4.3, the graphs $\Sigma_I(X)$ and $\Delta_I(X)$ are complete graphs. If $|X| \geq 8$, then both graphs $\Sigma_I(X)$ and $\Delta_I(X)$ have a subgraph that is isomorphic to K_8. Now, by Theorem 2.19, both graphs $\Sigma_I(X)$ and $\Delta_I(X)$ are not toroidal. Conversely, since K_8 has eight vertices, both graphs $\Sigma_I(X)$ and $\Delta_I(X)$ are not toroidal and so both graphs $\Sigma_I(X)$ and $\Delta_I(X)$ have at least eight vertices, which is contrary to the fact that $|X| \leq 7$. □

5 Graphs on the Ideals of KU-algebras Based on the Binary Operations $\wedge$

For this section, we assume that the set X represents a bounded commutative KU-algebra.

Definition 5.1. *Let I be an ideal of X. Then we construct a simple graph $\Upsilon_I(X)$ with vertex set X and two distinct vertices x and y are adjacent if and only if $x \wedge y \in I$.*

Lemma 5.2. *Let I be an ideal of X. Then $deg(x) = |X| - 1$ for all $x \in I$ in the graph $\Upsilon_I(X)$.*

Proof. Let $x \in I$ and let y be an arbitrary element of X. Then $(x \centerdot y) \centerdot y \in I$. Since $(x \centerdot y) \centerdot y \leq x$, as I is an ideal of X, $y \centerdot x \in E(\Upsilon_I(X))$. □

Theorem 5.3. *Let I be an ideal of X. Then the graph $\Upsilon_I(X)$ is regular if and only if it is complete.*

Proof. Let $\Upsilon_I(X)$ be a regular graph. By Lemma 5.2, we have $deg(1) = |X| - 1$. Since $\Upsilon_I(X)$ is regular, for any $x \in X, deg(x) = |X| - 1$. This means that $\Upsilon_I(X)$ is a complete graph. Conversely, a complete graph is always regular. □

The following Proposition 5.4 and Theorem 5.5 follow from Lemma 5.2.

Proposition 5.4. *Let I be an ideal of X. Then $\omega(Y_I(X)) \geq |I|$.*

Theorem 5.5. *Let I be an ideal of X. Then $\Upsilon_I(X)$ is connected and $diam(\Upsilon_I(X)) \leq 2$.*

Theorem 5.6. *Let I be an ideal of X. Then $gr(\Upsilon_I(X)) = 3$.*

Proof. Let $a \neq 1$ be an element in I and let x be an arbitrary element in X. Then it is easy to see that $1 - a - x - 1$ is a cycle of length 3 in $\Upsilon_I(X)$. □

Proposition 5.7. *Let I be an ideal of X. Then the following statements hold:*

1. *If $\Upsilon_I(X)$ is planar, then $|I| \leq 4$.*
2. *If $\Upsilon_I(X)$ is outer planar, then $|I| \leq 3$.*
3. *If $\Upsilon_I(X)$ is toroidal, then $|I| \leq 7$.*

Proof.

1. From Lemma 5.2, it follows that the graph $\Upsilon_I(X)$ is a complete graph on I. If $|I| \geq 5$, then $\Upsilon_I(X)$ has a subgraph isomorphic to K_5, By Kuratowski's theorem, the graph $\Upsilon_I(X)$ is not planar.

2. By Lemma 5.2, the graph $\Upsilon_I(X)$ is a complete graph on I. If $|I| \geq 4$, then $\Upsilon_I(X)$ has a subgraph isomorphic to K_4. By Definition 2.17, the graph $\Upsilon_I(X)$ is not outer planar.

3. Again by Lemma 5.2, the graph $\Upsilon_I(X)$ is a complete graph on I. If $|I| \geq 7$, then $\Upsilon_I(X)$ has a subgraph isomorphic to K_8. By Theorem 2.19, the graph $\Upsilon_I(X)$ is not toroidal.

□

Theorem 5.8. *For any ideal I of X if $\Upsilon_I(X)$ is an Euler graph, then $|X|$ is odd.*

Proof. According to Lemma 5.2, for all $x \in I$, the $deg(x) = |X|-1$. If $\Upsilon_I(X)$ is an Euler graph, then the degree of every vertex in I is even. Consequently, $|X|$ is odd. □

Theorem 5.9. *For any ideal I of X, if $I = \cap_{1\leq i\leq n} P_i$ and for each $1 \leq j \leq n$, the ideal $I \neq \cap_{1\leq i\leq n, i\neq j} P_i$, where P_i are prime ideals of X, then $\omega(Y_I(X)) = n = \chi(Y_I(X))$.*

Proof. For each j with $1 \leq j \leq n$, consider an element x_j in $(\cap_{1\leq i\leq n, i\neq j} P_i) - P_j$. We have $A = \{x_1, ..., x_n\}$ is a clique in $\Upsilon_I(X)$. Hence $\omega(\Upsilon_I(X)) \geq n$. Now, we prove that $\chi(\Upsilon_I(X)) \leq n$. Define a coloring f by putting $f(x) = min\{i; x \notin P_i\}$. Let $f(x) = k, x$ and y be adjacent vertices. So $x \notin P_k$ and $x \wedge y \in I$. Since P_k is prime, $y \in P_k$, and so $f(y) \neq k$. Now, since $\omega(\Upsilon_1(X)) \leq \chi(\Upsilon_I(X))$, the result holds. □

Theorem 5.10. *For any ideal I of X, if $I = \cap_{j\in J} P_j$, where P_j are prime ideals of X and J is an infinite set for each $i \in J$, also $I \neq \cap_{j\neq i} P_j$, then $\omega(\Upsilon_I(X)) = \infty = \chi(\Upsilon_I(X))$.*

Proof. For each $i \in J$, there exists $x_i \in (\cap_{j\neq i} P_j - P_i)$. It can be easily seen that the set of x_i forms an infinite clique in $\Upsilon_I(X)$. Since $\omega(\Upsilon_I(X)) \leq \chi(\Upsilon_I(X))$, the assertion holds. □

6 Conclusion

In this article we have studied and discussed ideal-annihilator, right-ideal-annihilator, left-ideal-annihilator for a KU-algebra. Moreover, construction of some main types of graphs in a bounded KU-algebra $(X, \centerdot, 0)$ based on ideals that are denoted by $\phi_1(X)$, $\Delta_1(X)$ *and* $\Sigma_1(X)$ were taken under consideration. Furthermore, basic graphical properties such as connectivity, regularity, and planarity on the structure of these graphs were investigated. Finally, we have constructed the graph $Y_I(X)$ and have studied its properties with these aspects.

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