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On the Diophantine equation $a^x + (a+2)^y = z^2$ where a is congruent to 19 modulo 28

Chokchai Viriyapong, Nongluk Viriyapong

Mathematics and Applied Mathematics Research Unit
Department of Mathematics
Mahasarakham University
Maha Sarakham, 44150, Thailand

email: chokchai.v@msu.ac.th, nongluk.h@msu.ac.th

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Abstract

In this article, we show that the Diophantine equation $a^x + (a+2)^y = z^2$ has no non-negative integer solution where $a \in \mathbb{Z}^+$ such that $a \equiv_{28} 19$.

1 Introduction

Many mathematicians studied the non-negative solutions (x, y, z) of Diophantine equations of the type $a^x + b^y = z^2$ where a and b are fixed. In 2020, Dokchann and Pakapongpun [1] proved that the Diophantine equation $a^x + (a+2)^y = z^2$ has no non-negative integer solution where a is a positive integer with $a \equiv_{42} 5$. Later, C. Viriyapong and N. Viriyapong [2] showed that the Diophantine equation $a^x + (a+2)^y = z^2$ has no non-negative integer solution where a is a positive integer with $a \equiv_{21} 5$.

In this paper, we study the Diophantine equation $a^x + (a+2)^y = z^2$ where a is a positive integer with $a \equiv_{28} 19$.

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2 Preliminaries

Throughout this article, $a \equiv_m b$ always means a and b are congruent modulo m where a, b, m are integers such that $m \geqslant 1$. For notational convenience, we will write $a \equiv_m b, c$ to mean that $a \equiv_m b$ or $a \equiv_m c$.

Now, we shall recall the Catalan's conjecture [3] of 1844, which was proved by Mihailescu [4] in 2004.

Theorem 2.1 (Catalan's conjecture). *The Diophantine equation* $a^x - b^y = 1$ *has a unique solution* $(a, b, x, y) = (3, 2, 2, 3)$ *where* a, b, x *and* y *are integers with* $\min\{a, b, x, y\} > 1$.

Next, we shall recall a lemma, which will be useful our work, in [2].

Lemma 2.2. *If* x *is a positive odd integer, then* $5^x \equiv_7 3, 5, 6$.

3 Main Results

Now, we shall discuss two important lemmas used in the main theorem.

Lemma 3.1. *Let* a *be a positive integer such that* $a \equiv_{28} 19$. *The Diophantine equation* $a^x + 1 = z^2$ *has no non-negative integer solution.*

Proof. Assume that there exist non-negative integers x and z such that $a^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$ which is a contradiction. Now, we have $x \geqslant 1$. Since $a \geqslant 19$, by Theorem 2.1, $x = 1$. Since $a \equiv_{28} 19$, $a \equiv_7 5$. Then $z^2 \equiv_7 6$ which contradicts the fact that $z^2 \equiv_7 0, 1, 2, 4$. The proof is complete. $\square$

Lemma 3.2. *Let* a *be a positive integer such that* $a \equiv_{28} 19$. *The Diophantine equation* $1 + (a + 2)^y = z^2$ *has no non-negative integer solution.*

Proof. Assume that there exist non-negative integers y and z such that $1 + (a + 2)^y = z^2$. If $y = 0$, $z^2 = 2$ which is impossible. Now, we have $y \geqslant 1$. Since $a \equiv_{28} 19$, $a + 2 \equiv_4 1$. Then $z^2 \equiv_4 2$. This contradicts the fact that $z^2 \equiv_4 0, 1$. This lemma is proved. $\square$

Next, we shall give our main result.

Theorem 3.3. *Let* a *be a positve integer such that* $a \equiv_{28} 19$. *The Diophantine equation* $a^x + (a + 2)^y = z^2$ *has no non-negative integer solution where* x, y, z *are non-negative integers.*

Proof. Assume that there exist non-negative integers x, y, z such that $a^x+(a+2)^y = z^2$. By Lemma 3.1 and 3.2, $x \geqslant 1$ and $y \geqslant 1$. If x is even, then $a^x \equiv_4 1$ because $a \equiv_{28} 19$. Since $(a+2)^y \equiv_4 1$, $z^2 \equiv_4 2$ which contradicts the fact that $z^2 \equiv_4 0, 1$. Now, we obtain that x is odd. Since $a \equiv_7 5$, by Lemma 2.2, we have $a^x \equiv_7 3, 5, 6$. Since $(a+2)^y \equiv_7 0$, we obtain $z^2 \equiv_7 3, 5, 6$, which contradicts the fact that $z^2 \equiv_7 0, 1, 2, 4$. This completes the proof. □

4 Conclusion

In this paper, we proved that the Diophantine equation $a^x + (a+2)^y = z^2$ has no non-negative integer solution where a is a positive integer such that $a \equiv_{28} 19$. Clearly, the Diophantine equations $47^x+49^y = z^2$ [5] and $131^x + 133^y = z^2$ [6] are two special cases of Theorem 3.3.

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