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A Note on Regularity of Transformation Semigroups with Fixed Sets

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Abstract

Let X be a nonempty set and let $T(X)$ denote the semigroup of transformations from X to itself under the composition of functions. For a fixed subset Y of a nonempty set X, let $Fix(X,Y)$ be the set of all self-maps on X which fix all elements in Y. Then $Fix(X,Y)$ is a regular subsemigroup of $T(X)$. In this paper, we characterize coregular elements of $Fix(X,Y)$ and give necessary and sufficient conditions for $Fix(X,Y)$ to be coregular. Moreover, we study some properties of regularity on $Fix(X,Y)$ and give necessary and sufficient conditions for the set of all left regular (right regular and completely regular) elements to be a subsemigroup of $Fix(X,Y)$.

1 Introduction

Regularity play an important role in the semigroup theory and they have been studied from various aspects. An element a in a semigroup S is said

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to be *regular* if $a = aba$ for some $b \in S$, *left regular* if $a = ba^2$ for some $b \in S$, *right regular* if $a = a^2b$ for some $b \in S$, *completely regular* if $a = aba$ and $ab = ba$ for some $b \in S$. In fact, a is both left and right regular if and only if a is completely regular. S is a *regular* [*left regular, right regular and completely regular*] *semigroup* if every element of S is regular [left regular, right regular and completely regular].

A special case of a regular element is a coregular element. An element a in a semigroup S is *coregular* if there exists $b \in S$ such that $a = aba = bab$, and S is *coregular* if every element in S is coregular. Clearly, an element a in a semigroup S is coregular if and only if $a^3 = a$, and every coregular element is regular, left regular, right regular and completely regular. Coregular semigroup was first introduced and studied by Bijev and Todorov [1].

We denote the set of all regular elements, left regular elements, right regular elements, completely regular elements and coregular elements of a semigroup S by $\mathrm{Reg}(S)$, $\mathrm{LReg}(S)$, $\mathrm{RReg}(S)$, $\mathrm{CReg}(S)$ and $\mathrm{CoReg}(S)$, respectively.

An element a in a semigroup S is said to be *idempotent* if $a^2 = a$. Then an idempotent of S is regular, left regular, right regular, completely regular and coregular. The set of all idempotents of S is denoted by $E(S)$.

Let X be a nonempty set, and $T(X)$ denote the set of all transformations from X to itself. Then $T(X)$ is a semigroup under the composition of maps and it is called the *full transformation semigroup* on X.

For a fixed subset Y of a nonempty set X, let

$$Fix(X,Y) = \{\alpha \in T(X) : y\alpha = y \text{ for all } y \in Y\}.$$

Then $Fix(X,Y)$ is a regular subsemigroup of $T(X)$. Note that $Fix(X,Y)$ contains the identity map on X, denoted by id_X. If $Y = \emptyset$, then $Fix(X,Y) = T(X)$. And if $X = Y$, then $Fix(X,Y)$ consists of one element, id_X.

The semigroup $Fix(X,Y)$ was first studied by Honyam and Sanwong [4] and they determined Green's relations, ideals and ranks of $Fix(X,Y)$. Later in 2015, Pookpienlert and Sanwong [5] gave necessary and sufficient conditions for elements of $Fix(X,Y)$ to be left regular, right regular, intra-regular and completely regular which directly obtain from Green's relations; and described the number of left regular elements. Recently, Chinram and Yonthanthum [2] gave necessary and sufficient conditions for the elements of $Fix(X,Y)$ to be left regular and right regular and applied these conditions to determine the left regularity and right regularity of $Fix(X,Y)$.

In this paper, we characterize coregular elements on $Fix(X,Y)$ and give necessary and sufficient conditions for $Fix(X,Y)$ to be coregular in Section

3. In Section 4, we characterize when LReg($Fix(X,Y)$), RReg($Fix(X,Y)$) and CReg($Fix(X,Y)$) to be subsemigroups of $Fix(X,Y)$.

2 Preliminaries

Throughout this paper, the cardinality of a set A is denoted by $|A|$. Also, we write functions on the right; in particular, this means that for a composition $\alpha\beta$, α is applied first.

Let X be a set and $\emptyset \neq Y \subsetneq X$. For convenience, in this paper, we let $Y = \{a_i : i \in I\}$, unless otherwise stated. Thus for each $\alpha \in Fix(X,Y)$, we have $a_i\alpha = a_i$ for all $i \in I$. So $Y = \{a_i : i \in I\} = Y\alpha \subseteq X\alpha$, the image of α. We modify the convention introduced in [3] vol. 2, p. 241 as below.

If $\alpha \in Fix(X,Y)$, then we write $\alpha = \begin{pmatrix} A_i & B_j \\ a_i & b_j \end{pmatrix}$ and take as understood that the subscripts i and j belong to the index sets I and J, respectively such that $X\alpha = \{a_i : i \in I\}\dot{\cup}\{b_j : j \in J\}$, $a_i\alpha^{-1} = A_i$, $b_j\alpha^{-1} = B_j$. Thus $A_i \cap Y = \{a_i\}$, $B_j \subseteq X \setminus Y$ and $\{b_j : j \in J\} \subseteq X \setminus Y$. Here, J can be an empty set.

It is well known that $\alpha \in T(X)$ is an idempotent if and only if $x\alpha = x$ for all $x \in X\alpha$. Consequently $\alpha \in Fix(X,Y)$ is an idempotent if and only if $x\alpha = x$ for all $x \in X\alpha \setminus Y$. It is easy to see that if $|X \setminus Y| = 1$, then all elements in $Fix(X,Y)$ are idempotents.

Left regularity and right regularity on $Fix(X,Y)$ were studied by Pookpienlert and Sanwong [5]; and Chinram and Yonthanthum [2] as shown in the following theorems.

Theorem 2.1. [5] *Let $\alpha \in Fix(X,Y)$. Then the following statements hold.*
(1) *α is left regular if and only if $X\alpha \setminus Y = X\alpha^2 \setminus Y$.*
(2) *α is right regular if and only if $\pi_\alpha = \pi_{\alpha^2}$ where $\pi_\alpha = \{x\alpha^{-1} : x \in X\alpha\}$.*

Theorem 2.2. *Let $\alpha \in Fix(X,Y)$ be such that $X\alpha \setminus Y$ is a finite set. Then α is left regular if and only if α is right regular.*

Theorem 2.3. *The number of left regular elements in $Fix(X,Y)$ is*
$$\sum_{m=0}^{n-r} \binom{n-r}{m} m!(r+m)^{n-r-m} \text{ where } |X| = n \text{ and } |Y| = r.$$

Theorem 2.4. [2] *If $|X| \leq 2$, then $Fix(X,Y)$ is a left (right) regular semigroup.*

Theorem 2.5. [2] *If $|X| > 2$, then $Fix(X,Y)$ is a left (right) regular semigroup if and only if $X = Y$ or $|X \setminus Y| = 1$.*

3 Coregular Elements on $Fix(X, Y)$

In this section, we characterize coregular elements of $Fix(X, Y)$ and give necessary and sufficient conditions for $Fix(X, Y)$ to be coregular. Also, we describe when CoReg($Fix(X, Y)$) is a subsemigroup of $Fix(X, Y)$.

Lemma 3.1. *If $|X| \leq 2$, then $Fix(X, Y)$ is coregular, left regular, right regular and completely regular.*

Proof. Assume that $|X| \leq 2$. If $X = Y$, then $Fix(X, Y) = \{id_X\}$ and hence $Fix(X, Y)$ is coregular, left regular, right regular and completely regular. If $Y = \emptyset$, then $Fix(X, Y) = T(X)$ and it is easy to prove that all elements in $Fix(X, Y)$ are coregular, left regular, right regular and completely regular. And, if $\emptyset \neq Y \subsetneq X$, then $|X \setminus Y| = 1$. Thus all elements in $Fix(X, Y)$ are idempotents and so each element is coregular, left regular, right regular and completely regular. $\square$

Theorem 3.2. *Let $\alpha \in Fix(X, Y)$. Then the following statements are equivalent.*

(1) *α is coregular.*
(2) *$x\alpha \in x\alpha^{-1}$ for all $x \in X\alpha \setminus Y$.*
(3) *$\alpha^2|_{X\alpha} = id_{X\alpha}$.*

Proof. Assume that α is coregular. Thus $\alpha^3 = \alpha$. Let $x \in X\alpha \setminus Y$. Then $x = z\alpha$ for some $z \in X \setminus Y$. So $x = z\alpha = z\alpha^3 = (x\alpha)\alpha$, that means $x\alpha \in x\alpha^{-1}$. Thus (1) $\Rightarrow$ (2). Now, to prove that (2) $\Rightarrow$ (3), assume that $x\alpha \in x\alpha^{-1}$ for all $x \in X\alpha \setminus Y$. Let $x \in X\alpha$. If $x \in Y \subseteq X\alpha$, then $x\alpha^2 = x = x\, id_{X\alpha}$. And, if $x \in X\alpha \setminus Y$, then $x\alpha^2 = (x\alpha)\alpha = x = x\, id_{X\alpha}$. Thus $\alpha^2|_{X\alpha} = id_{X\alpha}$. Assume that $\alpha^2|_{X\alpha} = id_{X\alpha}$. Let $x \in X$. Then $x\alpha^3 = (x\alpha)\alpha^2 = (x\alpha)\, id_{X\alpha} = x\alpha$ and so $\alpha^3 = \alpha$. Hence α is a coregular element. Therefore, (3) $\Rightarrow$ (1) as require. $\square$

As a consequence of Theorem 3.2, the necessary and sufficient conditions for the semigroup $Fix(X, Y)$ to be a coregular semigroup given as follows.

Theorem 3.3. *Let $X \geq 3$. Then $Fix(X, Y)$ is coregular if and only if $X = Y$ or $|X \setminus Y| = 1$.*

Proof. Assume that $X \neq Y$ and $|X \setminus Y| > 1$. We consider two cases.

Case 1: $Y = \emptyset$. Then $X \setminus Y = X$. Let x, y, z be distinct elements in X. Define $\alpha \in Fix(X, Y) = T(X)$ by $\alpha = \begin{pmatrix} x & y & X \setminus \{x, y\} \\ y & z & x \end{pmatrix}$. Then $y\alpha = z \notin \{x\} = y\alpha^{-1}$ and hence α is not coregular by Theorem 3.2.

Case 2: $Y \neq \emptyset$. Let $b, c \in X \setminus Y$ be such that $b \neq c$. Fix $i_0 \in I$ and let $I' = I \setminus \{i_0\}$ and $A = Y \setminus \{a_{i_0}\}$. Define $\alpha \in Fix(X,Y)$ by $\alpha = \begin{pmatrix} a_{i'} & b & X \setminus (A \cup \{b\}) \\ a_{i'} & c & a_{i_0} \end{pmatrix}$. Then $c\alpha = a_{i_0} \notin \{b\} = c\alpha^{-1}$. By Theorem 3.2, we get α is not coregular.

Conversely, assume that $X = Y$ or $|X \setminus Y| = 1$. If $X = Y$ then $Fix(X,Y) = \{id_X\}$. And if $|X \setminus Y| = 1$, then all elements in $Fix(X,Y)$ are idempotents, that means $\alpha^3 = \alpha$ for all $\alpha \in Fix(X,Y)$. Therefore, $Fix(X,Y)$ is a coregular semigroup. □

Theorem 3.4. *Let $|X| \geq 3$. Then* CoReg$(Fix(X,Y))$ *is a subsemigroup of $Fix(X,Y)$ if and only if $X = Y$ or $|X \setminus Y| = 1$.*

Proof. Assume that $X \neq Y$ and $|X \setminus Y| \neq 1$. We consider two cases.

Case 1: $Y = \emptyset$. Then $X \setminus Y = X$. Let x, y, z be distinct elements in X. Define $\alpha, \beta \in Fix(X,Y) = T(X)$ by

$$\alpha = \begin{pmatrix} \{x,y\} & X \setminus \{x,y\} \\ y & z \end{pmatrix} \text{ and } \beta = \begin{pmatrix} y & X \setminus \{y\} \\ y & x \end{pmatrix}.$$

Then α, β are idempotents and so α, β are coregular elements. We see that $\alpha\beta = \begin{pmatrix} \{x,y\} & X \setminus \{x,y\} \\ y & x \end{pmatrix}$ and $x \in X\alpha\beta$ such that $x(\alpha\beta) = y \notin X \setminus \{x,y\} = x(\alpha\beta)^{-1}$. By Theorem 3.2, $\alpha\beta$ is not a coregular element.

Case 2: $Y \neq \emptyset$. Let $b, c \in X \setminus Y$ be such that $b \neq c$. Fix $i_0 \in I$ and let $I' = I \setminus \{i_0\}$. Define $\alpha, \beta \in Fix(X,Y)$ by

$$\alpha = \begin{pmatrix} a_i & b & X \setminus (Y \cup \{b\}) \\ a_i & c & b \end{pmatrix} \text{ and } \beta = \begin{pmatrix} a_{i'} & \{b, a_{i_0}\} & X \setminus (Y \cup \{b\}) \\ a_{i'} & a_{i_0} & c \end{pmatrix}.$$

Then $\alpha^2|_{X\alpha} = id_{X\alpha}$ and β is an idempotent and so α and β are coregular. We get $\alpha\beta = \begin{pmatrix} a_{i'} & b & (X \setminus (Y \cup \{b\})) \cup \{a_{i_0}\} \\ a_{i'} & c & a_{i_0} \end{pmatrix}$ and $c(\alpha\beta) = a_{i_0} \notin \{b\} = c(\alpha\beta)^{-1}$. Thus $\alpha\beta$ is not a coregular element.

Conversely, assume that $X = Y$ or $|X \setminus Y| = 1$. By Theorem 3.3, $Fix(X,Y)$ is coregular and hence CoReg$(Fix(X,Y)) = Fix(X,Y)$ □

The following corollary is a direct consequence of Theorems 3.3 - 3.4.

Corollary 3.5. *Let $|X| \geq 3$. Then the following statements are equivalent.*

(1) *$Fix(X,Y)$ is coregular.*

(2) CoReg$(Fix(X,Y))$ *is a subsemigroup of $Fix(X,Y)$.*

(3) *$X = Y$ or $|X \setminus Y| = 1$.*

4 Some Properties of Regularity of $Fix(X,Y)$

This section, we give necessary and sufficient conditions for LReg($Fix(X,Y)$), RReg($Fix(X,Y)$) and CReg($Fix(X,Y)$) to be subsemigroups of $Fix(X,Y)$.

Theorem 4.1. *Let* $|X| \geq 3$. *Then the following statements are equivalent.*
(1) $Fix(X,Y)$ *is left regular.*
(2) LReg($Fix(X,Y)$) *is a subsemigroup of* $Fix(X,Y)$.
(3) $X = Y$ *or* $|X \setminus Y| = 1$.

Proof. From Theorem 2.5, we see that (1) $\Leftrightarrow$ (3). It is clear that (1) $\Rightarrow$ (2). Now to prove that (2) $\Rightarrow$ (3), assume that $X \neq Y$ and $|X \setminus Y| \neq 1$. If $Y = \emptyset$, then we define α, β as in Theorem 3.4 (Case 1). So α, β are idempotents and hence α, β are also left regular elements. We see that $(\alpha\beta)^2$ is a constant map with $X(\alpha\beta)^2 = \{y\}$, so $X(\alpha\beta) \setminus Y = \{x, y\} \neq \{y\} = X(\alpha\beta)^2 \setminus Y$. Thus $\alpha\beta$ is not left regular by Theorem 2.1 (1). If $Y \neq \emptyset$, then we define α, β as in Theorem 3.4 (Case 2). We see that $X\alpha \setminus Y = X\alpha^2 \setminus Y$ and β is an idempotent, that means α, β are also left regular elements. We obtain that $(\alpha\beta)^2 = \begin{pmatrix} a_{i'} & (X \setminus Y) \cup \{a_{i_0}\} \\ a_{i'} & a_{i_0} \end{pmatrix}$ and thus $X(\alpha\beta) \setminus Y = \{c\} \neq \emptyset = X(\alpha\beta)^2 \setminus Y$, which implies that $\alpha\beta$ is not left regular by Theorem 2.1 (1). Therefore, LReg($Fix(X,Y)$) is not a subsemigroup of $Fix(X,Y)$. □

Theorem 4.2. *Let* $|X| \geq 3$. *Then the following statements are equivalent.*
(1) $Fix(X,Y)$ *is right regular.*
(2) RReg($Fix(X,Y)$) *is a subsemigroup of* $Fix(X,Y)$.
(3) $X = Y$ *or* $|X \setminus Y| = 1$.

Proof. Obviously, (1) $\Rightarrow$ (2). From Theorem 2.5, we have (1) $\Leftrightarrow$ (3). Now, we prove that (2) $\Rightarrow$ (3). Assume that $X \neq Y$ and $|X \setminus Y| \neq 1$. Define $\alpha, \beta \in Fix(X,Y)$ as in Theorem 3.4. Then α, β are right regular elements. But, we obtain that $\pi_{\alpha\beta} \neq \pi_{(\alpha\beta)^2}$, thus $\alpha\beta$ is not right regular by Theorem 2.1 (2). So RReg($Fix(X,Y)$) is not a subsemigroup of $Fix(X,Y)$. □

As a direct consequence of Theorems 4.1 - 4.2, we obtain the following corollary.

Corollary 4.3. *Let* $|X| \geq 3$. *Then the following statements are equivalent.*
(1) $Fix(X,Y)$ *is completely regular.*
(2) CReg($Fix(X,Y)$) *is a subsemigroup of* $Fix(X,Y)$.
(3) $X = Y$ *or* $|X \setminus Y| = 1$.

We know that every coregular element is both left and right regular, but there are left and right regular elements which are not coregular as shown in the following lemma.

Lemma 4.4. *If* $|X \setminus Y| \geq 3$*, then there exists* $\alpha \in \mathrm{LReg}(Fix(X,Y)) \cap \mathrm{RReg}(Fix(X,Y))$ *such that* $\alpha \notin \mathrm{CoReg}(Fix(X,Y))$*.*

Proof. Assume that $|X \setminus Y| \geq 3$. Let a, b, c be distinct elements in $X \setminus Y$. Define $\alpha = \begin{pmatrix} a & b & c & x \\ b & c & a & x \end{pmatrix}_{x \in X \setminus \{a,b,c\}}$. Then $\alpha \in Fix(X,Y)$ and we see that $\alpha^2 = \begin{pmatrix} a & b & c & x \\ c & a & b & x \end{pmatrix}_{x \in X \setminus \{a,b,c\}}$. So $X\alpha \setminus Y = X \setminus Y = X\alpha^2 \setminus Y$ and $\pi_\alpha = \pi_{\alpha^2}$. By Theorem 2.1, we obtain that α is left regular and right regular. But, $\alpha \notin \mathrm{CoReg}(Fix(X,Y))$ since $c\alpha = a \notin \{b\} = c\alpha^{-1}$. $\square$

Recall that $\mathrm{CoReg}(Fix(X,Y)) = \mathrm{LReg}(Fix(X,Y)) = \mathrm{RReg}(Fix(X,Y)) = \mathrm{CReg}(Fix(X,Y)) = Fix(X,Y)$ when $|X| \leq 2$ or ($|X| \geq 3$, and $X = Y$ or $|X \setminus Y| = 1$). Here, there are some other cases that coregular elements, left regular elements, right regular elements and completely regular elements are coincide as the following theorem.

Theorem 4.5. *If* $(|X|, |Y|) \in \{(3,1),(4,2)\}$*, then* $\mathrm{CoReg}(Fix(X,Y)) = \mathrm{LReg}(Fix(X,Y)) = \mathrm{RReg}(Fix(X,Y)) = \mathrm{CReg}(Fix(X,Y))$*.*

Proof. Assume that $(|X|, |Y|) \in \{(3,1),(4,2)\}$. We consider two cases.

Case 1: $(|X|, |Y|) = (3,1)$. Let $X = \{a,b,c\}$ and $Y = \{a\}$. Then

$$E(Fix(X,Y)) = \left\{ id_X, \begin{pmatrix} a & \{b,c\} \\ a & b \end{pmatrix}, \begin{pmatrix} a & \{b,c\} \\ a & c \end{pmatrix}, \begin{pmatrix} \{a,b\} & c \\ a & c \end{pmatrix}, \begin{pmatrix} \{a,c\} & b \\ a & b \end{pmatrix}, \begin{pmatrix} \{a,b,c\} \\ a \end{pmatrix} \right\}.$$

Let $\alpha = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}$. Then α is coregular and so $E(Fix(X,Y)) \cup \{\alpha\} \subseteq \mathrm{CoReg}(Fix(X,Y)) \subseteq \mathrm{LReg}(Fix(X,Y))$. By Theorem 2.3, we get

$$|\mathrm{LReg}(Fix(X,Y))| = \sum_{m=0}^{2} \binom{2}{m} m!(1+m)^{2-m} = \binom{2}{0} 0!(1)^2 + \binom{2}{1} 1!(2)^1 + \binom{2}{2} 2!(3)^0 = 1+4+2 = 7 = |E(Fix(X,Y)) \cup \{\alpha\}|.$$ Thus $\mathrm{CoReg}(Fix(X,Y)) = \mathrm{LReg}(Fix(X,Y))$.

Case 2: $(|X|,|Y|)=(4,2)$. Let $X=\{a,b,c,d\}$ and $Y=\{a,b\}$. Then

$$E(Fix(X,Y))=\left\{id_X, \begin{pmatrix}\{a,c\} & b & d\\ a & b & d\end{pmatrix}, \begin{pmatrix}\{a,d\} & b & c\\ a & b & c\end{pmatrix}, \begin{pmatrix}a & \{b,c\} & d\\ a & b & d\end{pmatrix},\right.$$
$$\begin{pmatrix}a & \{b,d\} & c\\ a & b & c\end{pmatrix}, \begin{pmatrix}a & b & \{c,d\}\\ a & b & c\end{pmatrix}, \begin{pmatrix}a & b & \{c,d\}\\ a & b & d\end{pmatrix}, \begin{pmatrix}\{a,c\} & \{b,d\}\\ a & b\end{pmatrix},$$
$$\left.\begin{pmatrix}\{a,d\} & \{b,c\}\\ a & b\end{pmatrix}, \begin{pmatrix}\{a,c,d\} & b\\ a & b\end{pmatrix}, \begin{pmatrix}a & \{b,c,d\}\\ a & b\end{pmatrix}\right\}.$$

Define $\alpha=\begin{pmatrix}a & b & c & d\\ a & b & d & c\end{pmatrix}$. It is easy to check that $\alpha\in \mathrm{CoReg}(Fix(X,Y))$. By Theorem 2.3, we obtain that $|\mathrm{LReg}(Fix(X,Y))| = \sum_{m=0}^{2}\binom{2}{m}m!(2+m)^{2-m} = \binom{2}{0}0!(2)^2+\binom{2}{1}1!(3)^1+\binom{2}{2}2!(4)^0 = 12 = |E(Fix(X,Y))\cup\{\alpha\}|$. Hence $\mathrm{CoReg}(Fix(X,Y))=\mathrm{LReg}(Fix(X,Y))$.

Since X is a finite set and by Theorem 2.2, we have $\mathrm{CoReg}(Fix(X,Y)) = \mathrm{LReg}(Fix(X,Y)) = \mathrm{RReg}(Fix(X,Y)) = \mathrm{CReg}(Fix(X,Y))$. $\square$

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