International Journal of Mathematics and
Computer Science, **19**(2024), no. 2, 439–443

Geodetic Closure Polynomial of Graphs

Rosalio G. Artes Jr.[1], **Jeffrey Imer C. Salim**[1],
Regimar A. Rasid[1], **Jerrivieh I. Edubos**[2],
Bayah J. Amiruddin[1]

[1]Mindanao State University
Tawi-Tawi College of Technology and Oceanography
Sanga-Sanga, 7500 Bongao, Tawi-Tawi, Philippines

[2]Philippine Christian University
1648 Taft Ave, Malate, 1004 Metro Manila, Philippines

email: rosalioartes@msutawi-tawi.edu.ph,
jeffreyimersalim@msutawi-tawi.edu.ph, regimarrasid@msutawi-tawi.edu.ph,
jerrivieh.edubos.gsbm@pcu.edu.ph, bayahamiruddin@msutawi-tawi.edu.ph

(Received September 26, 2023, Accepted October 30, 2023,
Published November 10, 2023)

Abstract

In this paper, we introduced a new graph polynomial called the vertex geodetic closure polynomial of a graph and established results for paths, complete bipartite graphs, and graphs resulting from the join of two graphs.

1 Introduction

The study of graph polynomials surfaced in the field of discrete and applied mathematics contributing some applications in Chemistry, Biology, and Physics [2]. In 1994, Hoede and Li [4] introduced the independent set polynomial of graphs which counts the number of independent substructure of the vertex-set of a graph. In 2014, Laja and Artes [5] introduced the notion

Key words and phrases: Graph polynomial, vertex geodetic set, vertex geodetic polynomial.
AMS (MOS) Subject Classifications: 05C25, 05C30, 05C31.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

of convex subgraph polynomials where convex sets are the baseline substructure. Convexity in graphs uses the concept of geodetic closures. Vijayan and Dafini [6] defined the notion of geodetic polynomial and obtained the geodetic sets and polynomials of centipedes. Recently, Asdain, Salim, and Artes [1] introduced the geodetic bounds in graphs. Their paper established the maximum 2-closure vertex geodetic sets in a graph.

Being inspired by the previous works, we develop a new graph polynomial by considering 2-closure vertex geodetic sets. For graph theoretic concepts used in this study, the readers may refer to Harary [3].

For two vertices u and v in a graph G, the *geodetic closure* of $\{u, v\}$ is the set $I_G[u, v] = \{u, v\} \cup \{y : y$ lies in a u-v shortest path in $G\}$. A subset S of $V(G)$ is a 2-*closure vertex geodetic set* in G if there exists $(u, v) \in S \times S$ such that $I_G[u, v] = S$. The *vertex geodetic closure polynomial* of G is given by $g_v(G; x) = \sum_{i=1}^{\gamma_2(G)} g_i(G)x^i$, where $g_i(G)$ is the number of 2-closure vertex geodetic sets in G of cardinality i and $\gamma_2(G)$ is the cardinality of a maximal 2-closure vertex geodetic set in G. Note that for every $u \in V(G)$, $I_G[u, u] = \{u\}$. Hence, every singleton subset of $V(G)$ is a 2-closure vertex geodetic set in G. Also, if $uv \in E(G)$, then $I_G[u, v] = \{u, v\}$. Thus, every pair of adjacent vertices in G is a 2-closure vertex geodetic set in G. Consequently, $g_1(G) = |V(G)|$ and $g_2(G) = |E(G)|$.

2 Results

The following result characterizes the 2-closure vertex geodetic sets in P_n. Note that a subgraph of P_n is connected if and only if it is also a path. The following result immediately follows.

Lemma 2.1. *A subset S of $V(P_n)$ is a 2-closure vertex geodetic set in P_n if and only if $\langle S \rangle = P_r$ for some $r \in \{1, 2, 3, ..., n\}$.*

The next result establishes the vertex geodetic closure polynomial of paths.

Theorem 2.2. *For natural numbers $n \geq 1$, $g_v(P_n; x) = nx\mu(x) - x^2\mu'(x)$, where $\mu(x) = 1 + x + x^2 + \cdots + x^{n-2} + x^{n-1}$.*

Proof: Lemma 2.1 asserts that

$$\begin{aligned} g_v(P_n;x) &= nx + (n-1)x^2 + (n-2)x^3 + \cdots + 2x^{n-1} + x^n \\ &= nx[1 + x + x^2 + \cdots + x^{n-2} + x^{n-1}] \\ &\quad -x^2[1 + 2x + 3x^2 + \cdots + (n-2)x^{n-3} + (n-1)x^{n-2}] \\ &= nx\mu(x) - x^2\mu'(x), \end{aligned}$$

where $\mu(x) = 1 + x + x^2 + \cdots + x^{n-2} + x^{n-1}$. This completes the proof. □

Next, we characterize the 2-closure vertex geodetic sets of the complete bipartite graph $K_{m,n}$.

Lemma 2.3. *A nontrivial subset S of $V(K_{m,n})$ is a 2-closure vertex geodetic set in $K_{m,n}$ if and only if it satisfies one of the following conditions:*

(i) $S = \{u, v\}$, *where* $u \in V(\overline{K_m})$ *and* $v \in V(\overline{K_n})$

(ii) $S = V(\overline{K_n}) \cup \{w, z\}, w, z \in V(\overline{K_m})$

(iii) $S = V(\overline{K_m}) \cup \{c, d\}, c, d \in V(\overline{K_n})$

Proof: Let S be a 2-closure vertex geodetic set in $K_{m,n}$. Then $S = I_{K_{m,n}}[a, b]$ for some $\{a, b\} \subseteq V(K_{m,n}) = V(\overline{K_m}) \cup V(\overline{K_n})$. If $a \in V(\overline{K_m})$ and $b \in V(\overline{K_n})$, then $S = \{a, b\}$ and (i) is satisfied. If $\{a, b\} \subseteq V(\overline{K_m})$, then $S = I_{K_{m,n}}[a, b] = \{a, b\} \cup V(\overline{K_n})$ and (ii) is satisfied. Similarly, if $\{a, b\} \subseteq V(\overline{K_n})$, then $S = I_{K_{m,n}}[a, b] = \{a, b\} \cup V(\overline{K_m})$ and (iii) follows. The converse is clear taking the geodetic closures of $\{u, v\}, \{w, z\}$, and $\{c, d\}$. □

From the above lemma, we have the followng result on the vertex geodetic polynomial of the complate bipartite graph $K_{m,n}$.

Theorem 2.4. *For natural numbers $m, n \geq 2$, $g(K_{m,n}, x) = (m + n)x + mnx^2 + \binom{n}{2}x^{m+2} + \binom{m}{2}x^{n+2}$.*

Proof: The first and second terms follow from the order and size of $K_{m,n}$, respectively. Now, for every pair of vertices $\{w, z\} \subseteq V(\overline{K_m})$, $I_{K_{m,n}}[w, z] = \{w, z\} \cup V(\overline{K_n})$. Similarly, for $\{c, d\} \subseteq V(\overline{K_n})$, $I_{K_{m,n}}[c, d] = \{c, d\} \cup V(\overline{K_m})$. Consequently, $g_{m+2}(K_{m,n}) = \binom{n}{2}$ and $g_{n+2}(K_{m,n}) = \binom{m}{2}$. The polynomial follows. □

The following lemma characterizes the 2-closure vertex geodetic sets in the join $G \oplus H$.

Lemma 2.5. *Let G and H be nontrivial connected graphs. A subset S of $V(G \oplus H)$ with $|S| \geq 3$ is a 2-closure vertex geodetic set in $G \oplus H$ if and only if it satisfies one of the following:*

(*i*) *$S = S_G \cup V(H)$, where S_G is a 2-closure vertex geodetic set in G of diameter 2.*

(*ii*) *$S = S_H \cup V(G)$, where S_H is a 2-closure vertex geodetic set in H of diameter 2.*

(*iii*) *$V(H) \cup \{u, v\}$, where $d_G(u, v) > 2$.*

(*iv*) *$V(G) \cup \{w, z\}$, where $d_H(u, v) > 2$.*

Proof: Assume that S is a 2-closure vertex geodetic set in G. Then there exist $a, b \in V(G \oplus H)$ such that $I_{G\oplus H}[a, b] = S$. Note that if $a \in V(G)$ and $b \in V(H)$, then $I_{G\oplus H}[a, b] = \{a, b\}$. Hence, we consider only when either $\{a, b\} \subseteq V(G)$ or $\{a, b\} \subseteq V(H)$ to have $|S| \geq 3$.

Case 1: $\{a, b\} \subseteq V(G)$.

Subcase 1.1: $dist_G(a, b) = 2$.
If $dist_G(a, b) = 2$, then the diameter of $I_G[a, b]$ is 2. Moreover, $I_{G\oplus H}[a, b] = I_G \cup V(H)$. This gives condition (*i*).

Subcase 1.2: $dist_G(a, b) > 2$
In this case, $I_{G\oplus H}[a, b] = \{a, b\} \cup V(H)$. This gives condition (*iii*).

Case 2: $\{a, b\} \subseteq V(H)$.

Subcase 2.1: $dist_H(a, b) = 2$. This is similar to Subcase 1.1 which implies condition (*ii*).

Subcase 2.2: $dist_H(a, b) > 2$
In this case, $I_{G\oplus H}[a, b] = \{a, b\} \cup V(G)$. This gives condition (*iv*).

The converse is clear by taking the closures of $\{a, b\}$, $\{u, v\}$, and $\{w, z\}$. The proof is complete. □

Denote by $S_p(G, diam_2)$ a 2-closure vertex geodetic set of G of diamater 2 with cardinality p. Denote by $\alpha(G, dist_{>2})$ the number of pairs of vertices in G with distance greater than 2.

The vertex geodetic polynomial of the graph resulting from the join of two nontrivial connected graphs G and H is established in the following result.

Theorem 2.6. *Let G and H be nontrivial connected graphs. Then*

$$\begin{aligned} g_v(G \oplus H; x) &= (|V(G)| + |V(H)|)x + (|E(G)| + |E(H)|)x^2 \\ &\quad + |V(G)||V(H)|)x^2 + \sum_p x^{|S_p(G, diam_2)| + |V(H)|} \\ &\quad + \sum_q x^{|S_q(H, diam_2)| + |V(G)|} + \alpha(G, dist_{>2})x^{2+|V(H)|} \\ &\quad + \alpha(H, dist_{>2})x^{2+|V(G)|}. \end{aligned}$$

Proof: The first three terms follow from the order and size of $G \oplus H$. The 4th, 5th, 6th and 7th terms follow from Lemma 2.5 (i), (iii), (ii), and (iv), respectively. □

References

[1] A. M. Asdain, J. I. C. Salim, R. G. Artes Jr., *Geodetic Bounds in Graphs*, *International Journal of Mathematics and Computer Science*, **18**, no. 4, (2023), 767–771.

[2] J. Ellis-Monaghan, J. Merino, *Graph Polynomials and Their Applications II: Interrelations and Interpretations*, Birkhauser, Boston, 2011.

[3] F. Harary, *Graph Theory*, CRC Press, Boca Raton, 2018.

[4] C. Hoede, X. Li, Clique polynomials and independent set polynomials of graphs, *Discrete Mathematics*, **125,** (1994), 219–228.

[5] L. S. Laja, R. G. Artes Jr., Zeros of Convex Subgraph Polynomials. *Applied Mathematical Sciences*, **8,** no. 59, (2014), 2917–2923. http://dx.doi.org/10.12988/ams.2014.44285

[6] A. Vijayan, T. Binu Selin, On Total Edge Fixed Geodominating Sets and Polynomials of Graphs, *International Journal of Mathematics*, **3,** no. 2, (2012), 1495–1501.