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Injective Domination Polynomial of the Dipyramidal Graphs

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Abstract

In this paper, we establish the injective domination polynomial of n-dipyramidal graph.

1 Introduction

The study of domination in graph is a fast growing area in discrete mathematics and it came as a result of the study of games such as chess where the goal is to dominate various squares of a chessboard by certain chess pieces. The concept of domination is used by De Jaenisch in 1862 while studying the problems of determining the minimum number of queens to dominate chessboard. The fastest growth in the study of dominating set-in graph theory began in 1960. Later the concept of domination set and domination number was used by Ore in 1962, Cockayne and Hedetniemi in 1977.

There are many applications of domination in graphs. The concept of domination helps in finding the shortest route (i.e., locating the shortest

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route to reach a hospital for a person from his location) and longest route (i.e., locating the hospital in such a place so that everyone can be serviced). Other applications include land surveying, Electrical Networks, Networking, Routing problems, Nuclear plants problems, Modelling Problems, and Coding Theory.
In this paper, we study the injective domination polynomial of an n-Dipyramidal graph. The neighborhood of vertices is considered in measuring the domination polynomial.

Let G be a graph. The *injective domination polynomial* of G, denoted by $D_{in}(G,x)$, is

$$D(G,x) = \sum_{i=\gamma(G)}^{|v(G)|} d(G,i)x^i,$$

where $\gamma(G)$ is the domination number of G and $d_{in}(G,i)$ is the number of injective set of G of size i .

Consider the cycle graph of order 4. C_4 has one injective dominating set of size four and four injective dominating sets of sizes three and two. Then the injective domination polynomial of C_4 is $D_{in}(C_4,x) = x^4 + 4x^3 + 4x^2$.

2 Main Result

An n-dipyramidal graph, denoted by dP_n, is the skeleton of an n-sided dipyramid. It is isomorphic to the $(m,2)$-cone graph $C_n \oplus \overline{K_2}$, where C_n is a cycle of order n and $\overline{K_2}$ is an empty graph of order 2. The n-dipyramidal graph has a vertex count of $n+2$ and an edge count of $3n$.

The following result gives the injective domination polynomial of the n-dipyramidal graph.

Theorem 2.1. *For $n \geq 3$,*

$$D_{in}(dP_n,x) = (1+x)^{n+2} - 1$$

Proof. Let $V(dP_n) = \{v_1, v_2, v_3, ..., v_n, v_{n+1,}v_{n+2}\}$. Let S_1 be a subset of $V(dP_n)$. The following shows the injective dominating sets of n-dipyramidal graph by S_1 and the number of injective dominating sets.
$S_{1(dP_3)} = \{v_1, v_2, v_3, v_4, v_5\}$, $d_{in}(dP_3,x) = \binom{3+2}{1}x$.
So the injective domination polynomial for the set of cardinality 1 is

$5x = \binom{5}{1}x;$

$S_{1(dP_4)} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $d_{in}(dP_4, 1) = 6x = \binom{4+2}{1}x.$
So the injective domination polynomial for the set of cardinality 1 is
$6x = \binom{6}{1}x;$

$S_{1(dP_5)} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, $d_{in}(dP_5, 1) = 7x\binom{5+2}{1}x.$
So the injective domination polynomial for the set of cardinality 1 is
$7x\binom{7}{1}x;$

$\vdots$

$S_{1(dP_n)} = \{v_1, v_2, v_3, v_n, ...v_{n+1}, v_{n+2}\}$, $d_{in}(dP_n, 1) = (n+2)x^r = \binom{n+2}{1}x^r.$
As a result, we have the term $\binom{n+2}{1}x^r$.
Next, if we add a vertex in every injective dominating sets of n-Dipyramidal denoted by $S_2 \subseteq V(dP_n)$, we have the following:
$S_{2(dP_3)} = \{\ \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}$
$, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}\}$, $d_{in}(dP_3, 2) = 10x^2 = \binom{3+2}{2}x^2.$
So the injective domintion polynomial for the of cardinality 2 is
$d_{in}(dP_3, 2) = 10x^2 = \binom{5}{2}x^2;$
$S_{2(dP_4)} = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_1, v_6\}, \{v_2, v_3\}, \{v_2, v_4\},$
$\{v_2, v_5\}, \{v_2, v_6\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_3, v_6\}, \{v_4, v_5\}, \{v_4, v_6\}, \{v_5, v_6\}\}$, $d_{in}(dP_4, 2)$
$= 15x^2 = \binom{4+2}{2}x^2.$
So the injective domination polynomial for the set of cardinality 2 is
$d_{in}(dP_4, 2) = 15x^2 = \binom{6}{2}x^2;$
$S_{2(dP_5)} = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_1, v_6\}, \{v_1, v_7\}, \{v_2, v_3\},$
$\{v_2, v_4\}, \{v_2, v_5\}, \{v_2, v_6\}, \{v_1, v_7\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_3, v_6\}, \{v_3, v_7\}, \{v_4, v_5\}, \{v_4, v_6\},$
$\{v_4, v_7\}, \{v_5, v_6\}, \{v_5, v_7\}, \{v_6, v_7\}\}.$

Thus $d_{in}(dP_5, 2) = 21x^2 = \binom{5+2}{2}x^2.$
So the injective domination polynomial for the set of cardinality 2 is
$d_{in}(dP_5, 2) = 21x^2 = \binom{7}{2}x^2;$

$\vdots$

$S_{2(dP_n)} = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, ..., \{v_1, v_{n+1}\}, \{v_1, v_{n+2}\}, \{v_2, v_3\},$
$\{v_2, v_4\}, \{v_2, v_5\}, ..., \{v_2, v_{n+1}\}, \{v_2, v_{n+2}\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_3, v_6\}, ..., \{v_3, v_{n+1}\}$
$\{v_3, v_{n+2}\}, ..., \{v_{n+1}, v_{n+2}\}\}$.

This implies that $d_{in}(dP_n, 2) = x^r = \binom{n+2}{n+2} x^r$.

Finally, the maximum injective dominating set of dP_n is $S_n = V(dP_n)$. We have the following:
$S_{n(dP_3)} = \{\{v_1, v_2, v_3, v_4, v_5\}\}$.

Hence $d_{in}(dP_n, 5) = x^5 = \binom{5}{5}$.

So the injective domination polynomial for the set of cardinality 5 is
$x^5 = d_{in}(dP_n, 5) = \binom{5}{5} x^5$;

$S_{n(dP_4)} = \{\{v_1, v_2, v_3, v_4, v_5, v_6\}\}$, $d_{in}(dP_n, 6) = x^6 = \binom{6}{6}$.
So the injective domination polynomial for the cardinality 6 is
$x^6 = \binom{6}{6} x^6$;

$S_{n(dP_5)} =\{\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}\}$, $d_{in}(dP_n, 7) = x^7 = \binom{7}{7}$.
So the injective domination polynomial for the set of cardinality 7 is
$x^7 = \binom{7}{7} x^7$

$\vdots$

$S_{n(dP_n)} = \{\{v_1, v_2, v_3, ..., v_{n+1}, v_{n+2},\}\}$, $d_{in}(dP_n, n) =1 = \binom{n+2}{n+2}$.
So the injective domination polynomial for the set of cardinality $n + 2$ is
$x^{n+2} = \binom{n+2}{n+2} x^{n+2}$.

Accordingly,

$$\begin{aligned} D_{in}(dP_n, x) &= \binom{n+2}{1}x + \binom{n+2}{2}x^2 + \binom{n+3}{3}x^3, ..., + \binom{n+2}{n+1}x^{n+1} \\ &+ \binom{n+2}{n+2}x^{n+2} \\ &= \sum_{r=1}^{n+2}\binom{n+2}{r}x^r \\ &= (1+x)^{n+2} - 1. \end{aligned}$$

This proves the theorem. □

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