International Journal of Mathematics and
Computer Science, **19**(2024), no. 2, 357–363

Analytical Solution for Time Fractional Reaction-Diffusion-Convection Model

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(Received September 1, 2023, Accepted October 2, 2023,
Published November 10, 2023)

Abstract

In this paper, we employ an algorithm that is based on multiple fractional power series (MFPS) to tackle the time fractional reaction-diffusion-convection (TF-RDC) model in Caputo sense. We construct a rapidly convergent series by minimizing the residual function and derive the coefficients of the MFPS through a chain of successive equations with lower cost and minimum efforts.
The effect of Caputo operator to TF-RDC is evident from the obtained solution curves for different fractional orders. Values of residual functions are tabulated to prove the accuracy of our algorithm.

1 Introduction

In the last few decades, many real problems were translated into mathematical models via fractional differential equations (FDEs) due to its ability to keep not only the original behavior of physical systems, but also their historical states [1]. One of these models is the reaction-diffusion-convection

Key words and phrases: Fractional Caputo derivative, Multiple residual power series, Reaction-Diffusion-Convection Model.
AMS (MOS) Subject Classifications: 35A22, 26A33, 35K57.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

(RDC) model which is a nonlinear partial differential equation (PDE) that studies the effect of three factors; reaction, convention and diffusion; to the concentration of the substance distributed. It is applicable in many fields of science, such as in heat conduction [2] and in cardiac arrhythmias [3]. It can be formulated as

$$ {}_t^*D_0^\beta Q(x,t) = (\Psi(Q)Q_x)_x + \Phi(Q)Q_x + \Upsilon(Q), \quad Q(x,0) = B(x) \tag{1.1} $$

such that $t \geq 0, x \in \mathbb{R}, \beta \in (0,1], Q = Q(x,t)$ is an unknown function to be determined, the functions Ψ, Φ and Υ denote the diffusion, the convention, and the reaction terms, respectively. While ${}_t^*D_0^\beta$ is the Caputo time fractional derivative which is defined as follows.

Definition 1.1. *[1] Let $\beta \in (0,1]$. The Caputo fractional derivative of order β for $Q(t)$ is defined by*

$$ {}_t^*D_a^\beta Q(t) = \begin{cases} \dfrac{1}{\Gamma(1-\beta)} \int_a^t (t-\eta)^{-\beta} Q'(\eta) d\eta, & \beta \in (0,1) \\ Q'(t), & \beta = 1 \end{cases} $$

provided the right hand side exists point wise on $[a,b]$.

Unfortunately, dealing with FDE's and getting their exact or even analytic solutions is not an easy task. Consequently, several powerful techniques have been modified to approximate their solutions. Examples of such techniques can be found in [4-5] and the references therein. One useful technique is the fractional residual power series method (FRPSM) which was suggested in [6] and was rapidly applied to many FDE's [7-8]. In order to apply the FRPSM for solving TF-RDCE, we assume the solution has MFPS which is defined as follows.

Definition 1.2. *[9] Let $\beta \in (0,1]$ and $t \in [a,\infty)$. The MFPS about a has the form*

$$ \sum_{m=0}^{\infty} \gamma_m(x)(t-a)^{\beta m} = \gamma_0(x) + \gamma_1(x)(t-a)^\beta + \gamma_2(x)(t-a)^{2\beta} + \cdots . \tag{1.2} $$

2 The FRPS Algorithm to Solve TF-RDCM

To apply the FRPSM to solve the TF-RDCM, we have the following algorithm.

Step (1) Assume the solution of (1.1) has the MFPS form

$$Q(x,t) = \sum_{m=0}^{\infty} \frac{\gamma_m(x) t^{m\beta}}{\Gamma(\beta m + 1)}$$

and define the residual function by

$$Resid_Q(x,t) =_t^* D_0^\beta Q(x,t) - (\Psi(Q)Q_x)_x - \Phi(Q)Q_x - \Upsilon(Q).$$

Step (2) Find the 0-th coefficient using the initial condition $Q(x,0) = B(x)$.
Step (3) Define the approximate solution as the N-th truncated MFPS of Q:

$$Q_N(x,t) = B(x) + \sum_{m=1}^{N} \frac{\gamma_m(x) t^{m\beta}}{\Gamma(\beta m + 1)}.$$

Step (4) Define the N-th residual function as

$$Resid_{Q,N}(x,t) =_t^* D_0^\beta Q_N(x,t) - (\Psi(Q_N)(Q_N)_x)_x - \Phi(Q_N)(Q)_{N_x} - \Upsilon(Q_N).$$

Step (5) Manipulate the equations ${}_t^*D_0^{(k-1)\beta} Resid_{Q,N}(x,0) = 0, k = 1, 2, 3, \ldots, N$.
Step (6) Find the coefficient $\gamma_N(x)$ by solving

$${}_t^*D_0^{(N-1)\beta} Resid_{Q,N}(x,0) = 0.$$

Step(7) Achieve the required accuracy by repeating Steps (3-6).

3 Numerical Applications

Example 3.1 Consider the TF-RDC equation:

$${}_t^*D_0^\beta Q(x,t) = Q_{xx}(x,t) + Q(x,t)(Q_x(x,t) + 1 - Q(x,t)), Q(x,0) = 1 + e^x \quad (3.3)$$

Following the FRPSM algorithm with $N = 10$, we obtain $\gamma_N(x) = e^x, N = 1, 2, \ldots, 10$. So, the 10th truncated MFPS is $Q_{10}(x,t) = 1 + e^x \sum_{k=0}^{10} \dfrac{t^{\beta k}}{\Gamma(k\beta + 1)}$. Hence, we may conclude that the N-th truncated MFPS has the form:

$$Q_N(x,t) = 1 + e^x \sum_{k=0}^{N} \frac{t^{\beta k}}{\Gamma(k\beta + 1)}, \quad (3.4)$$

and the exact solution is

$$Q(x,t) = 1 + e^x \sum_{k=0}^{\infty} \frac{t^{\beta k}}{\Gamma(k\beta + 1)} = 1 + e^x E_\beta(t^\beta), \tag{3.5}$$

where $E_\beta(t^\beta)$ is the Mittag-Leffler function [1]. To check our assumption in (3.2), we compute $Resid_Q(x,t)$ which must be zero. In fact,

$$\begin{aligned} Resid_Q(x,t) &= {}_t^* D_0^\beta Q(x,t) - Q_{xx}(x,t) - Q(x,t)(Q_x(x,t) + 1 - Q(x,t)) \\ &= {}_t^* D_0^\beta (1 + e^x E_\beta(t^\beta)) - \frac{d^2}{dx^2}(1 + e^x E_\beta(t^\beta)) \\ &- (1 + e^x E_\beta(t^\beta))(\frac{d}{dx}(1 + e^x E_\beta(t^\beta)) + 1 - (1 + e^x E_\beta(t^\beta))) = 0. \end{aligned}$$

The surface plots for $\beta \in \{0.8, 0.6\}$ when $t \in [0,4]$ and $x \in [-4,4]$ are presented in Figure 1.

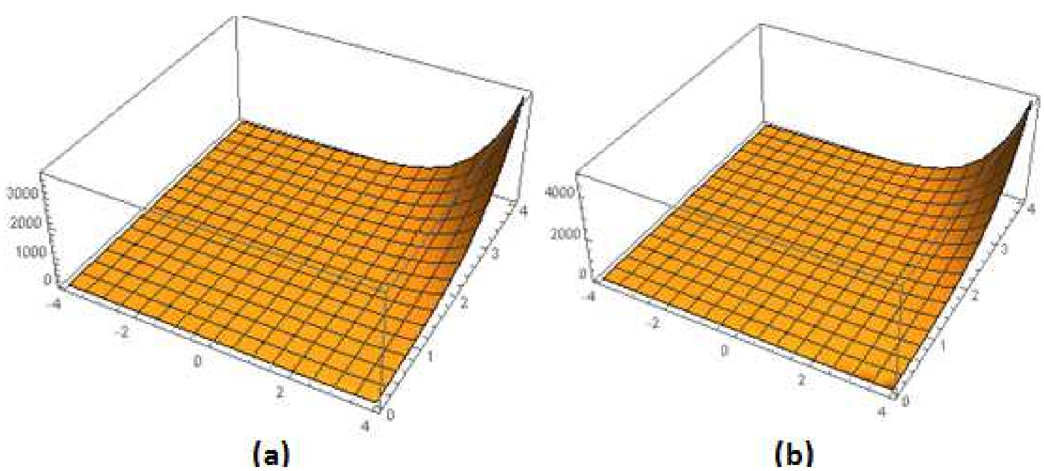

Figure 1: The surface plots for the analytic solutions with (a) $\beta = 0.8$ and (b) $\beta = 0.6$ for example 3.1.

Example 3.2 Consider the TF-RDC equation:

$$\begin{gathered} {}_t^* D_0^\beta Q(x,t) = (Q(x,t)Q_x(x,t))_x + Q(x,t)(3Q_x(x,t) + 2 - 2Q(x,t)), \quad (3.6) \\ Q(x,0) = 2\sqrt{e^x - e^{-4x}}. \end{gathered}$$

Using the FRPSM, we obtain:

$$\begin{aligned} &\gamma_0(x) = 2\sqrt{e^x - e^{-4x}}, \quad \gamma_1(x) = 4\sqrt{e^x - e^{-4x}}, \quad \gamma_2(x) = 8\sqrt{e^x - e^{-4x}}, \\ &\gamma_3(x) = 16\sqrt{e^x - e^{-4x}}, \quad \gamma_4(x) = 32\sqrt{e^x - e^{-4x}}, \quad \gamma_5(x) = 64\sqrt{e^x - e^{-4x}} \end{aligned}$$

So, the fifth truncated power series of (3.3) is

$$Q_5(x,t) = 2\sqrt{e^x - e^{-4x}}(1 + \frac{2t^\beta}{\Gamma(\beta+1)} + \frac{4t^{2\beta}}{\Gamma(2\beta+1)} + \frac{8t^{3\beta}}{\Gamma(3\beta+1)} + \frac{16t^{4\beta}}{\Gamma(4\beta+1)} + \frac{32t^{5\beta}}{\Gamma(5\beta+1)}) = 2\sqrt{e^x - e^{-4x}} \sum_{k=0}^{5} \frac{(2t^\beta)^k}{\Gamma(k\beta+1)}.$$

Hence, we may suggest the N-th truncated MFPS as

$$Q_N(x,t) = 2\sqrt{e^x - e^{-4x}} \sum_{k=0}^{N} \frac{(2t^\beta)^k}{\Gamma(k\beta+1)},$$

and the exact solution as

$$Q(x,t) = 2\sqrt{e^x - e^{-4x}} \sum_{k=0}^{\infty} \frac{(2t^\beta)^k}{\Gamma(k\beta+1)} = 2\sqrt{e^x - e^{-4x}} E_\beta(2t^\beta).$$

The FRPSM converges rapidly to the exact solution as it is evident from Figure 2 that represents the N-truncated MFPS for different number of iterations at $x = 1$ and $\beta = 0.75$. Also, some tabulated values of $Resid_{Q,20}(2,t)$ are listed in Table 1. On the other hand, we display the solution curves when $\beta \in \{1, 0.95, 0.85, 0.75\}$ and $x = 1$ in Figure 3 to notice the effect of Caputo fractional operator to the TF-RDC in (3.3).

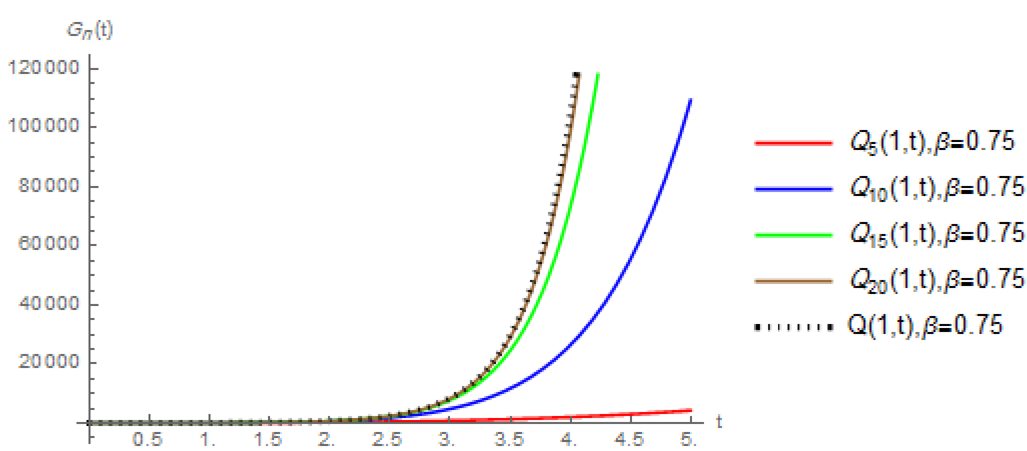

Figure 2: The Nth FRPS solution at $x = 1$ for example 3.2.

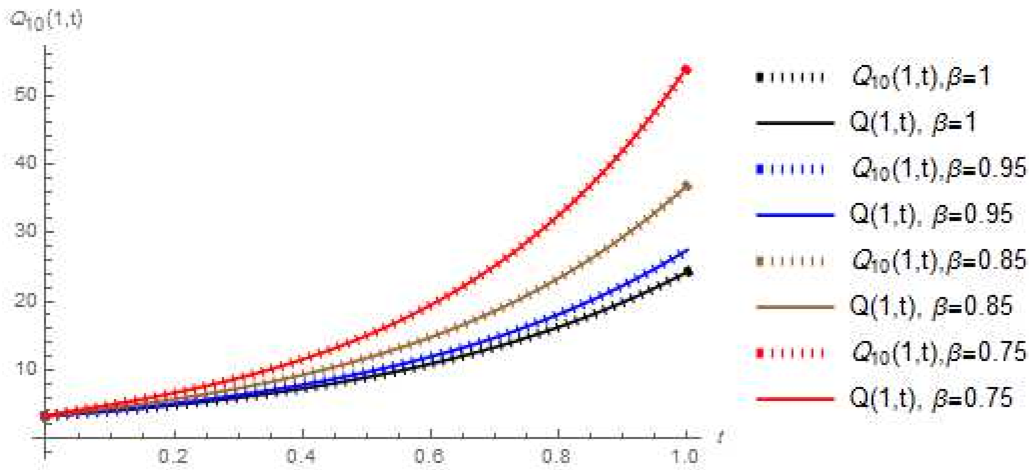

Figure 3: Comparison between the exact and approximate solutions at $x = 1$ for example 3.2.

Table 1: The 20-th residual errors at x=1 for example 3.2

t	$\beta = 0.95$	$\beta = 0.85$	$\beta = 0.75$
0.0	0.0	0.0	0.0
0.1	-5.66558×10^{-30}	-1.93763×10^{-25}	-5.27035×10^{-21}
0.2	-2.9704×10^{-24}	-2.53969×10^{-20}	-1.72699×10^{-16}
0.3	-6.58489×10^{-21}	-2.50226×10^{-17}	-7.56238×10^{-14}
0.4	-1.55734×10^{-18}	-3.32882×10^{-15}	-5.6590×10^{-12}
0.5	-1.08062×10^{-16}	-1.47829×10^{-13}	-1.60838×10^{-10}
0.6	-3.45238×10^{-15}	-3.27976×10^{-12}	-2.47804×10^{-9}
0.7	-6.45814×10^{-14}	-4.50752×10^{-11}	-2.50213×10^{-8}
0.8	-8.16496×10^{-13}	-4.36315×10^{-10}	-1.85434×10^{-7}
0.9	-7.65336×10^{-12}	-3.23142×10^{-9}	-1.08512×10^{-6}
1.0	-5.66558×10^{-11}	-1.93763×10^{-8}	-5.27035×10^{-6}

4 Conclusion

In this paper, we apply an analytic method to solve TF-RDC. The proposed algorithm proved its efficiency to get MFPS solutions that rapidly converge to the exact solutions with less effort and times. Its power appeared in its success in obtaining the exact solutions for this type of nonlinear fractional PDEs.

The effect of the Caputo fractional derivative on TF-RDC was obvious through plotting the solution curves for different values of the fractional order. These curves approach the solution of classical integer order RDC as the fractional order approaches the integer order.

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