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On the Diophantine Equation $\frac{1}{a}+\frac{1}{b}=\frac{m}{pq}$

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Abstract

In this article, we find all positive integer solutions of the Diophantine Equation $\frac{1}{a}+\frac{1}{b}=\frac{m}{pq}$, where m is a positive integer and p, q are distinct prime numbers.

1 Introduction

Let m be a positive integer and p, q be distinct prime numbers. Recently, researchers have been interested in studying all positive integer solutions of the Diophantine equation in the form

$$\frac{1}{a}+\frac{1}{b}=\frac{m}{pq}. \tag{1.1}$$

For example, in 2022, Johnson [1] found all positive integer solutions of the equation (1.1), where $m=q+1$ with $(q-1) \mid (p+1)$. After that, Prugsapitak ([2],[3]) considered the case $m=q-1$ with $p>q$. In this paper, we investigate all positive integer solutions of the equation (1.1) for any positive integer m using only elementary methods.

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2 Main Results

Throughout this paper, we assume that m is a positive integer and p, q are distinct prime numbers. First, we find all positive integer solutions (a, b) of the equation (1.1), when $m = 1$.

Lemma 2.1. *If $m = 1$, then all positive integer solutions of (1.1) are*

$$(a, b) \in \{(p(p+q), q(p+q)), (p(q+1), pq(q+1)), (pq(p+1), q(p+1)), (2pq, 2pq), (pq(pq+1), pq+1)\}.$$

Proof. Let (a, b) be a positive integer solution of (1.1). Then $(a+b)pq = ab$. Since p is prime, we have $p \mid a$ or $p \mid b$. Without loss of generality, we may assume that $p \mid a$. Then $a = pa_1$, for some positive integer a_1. It implies that $(pa_1 + b)q = a_1 b$. Therefore, $q \mid a_1 b$.

Case 1. $q \mid b$. Then there exists a positive integer b_1 such that $b = qb_1$. Thus $qb_1 = a_1(b_1 - p)$ from which it follows that $a_1 \mid qb_1$. Since q is prime, we consider the following two subcases.

Subcase 1.1 $\gcd(q, a_1) = 1$. Then $a_1 \mid b_1$ and so $b_1 = a_1 b_2$, for some positive integer b_2. Thus $p = (a_1 - q)b_2$. If $p = a_1 - q$ and $b_2 = 1$, then $a_1 = p+q$ and $b_1 = p+q$. Therefore, $(a, b) = (p(p+q), q(p+q))$. If $p = b_2$ and $a_1 - q = 1$, then $a_1 = q+1$ and $b_1 = p(q+1)$. Hence $(a, b) = (p(q+1), pq(q+1))$.

Subcase 1.2 $\gcd(q, a_1) = q$. Then $q \mid a_1$ and so $a_1 = qa_2$, for some positive integer a_2. It implies that $pa_2 = (a_2 - 1)b_1$. Since p is prime, we get $\gcd(p, b_1) = 1$ or p. Assume that $\gcd(p, b_1) = 1$. Since $\gcd(a_2, a_2 - 1) = 1$, we obtain $p = a_2 - 1$ and $a_2 = b_1$. Thus $a_2 = p+1, b_1 = p+1$ and $a_1 = q(p+1)$. Then $(a, b) = (pq(p+1), q(p+1))$. If $\gcd(p, b_1) = p$, then $p \mid b_1$ and so $b_1 = pb_2$, for some positive integer b_2. Consequently, $a_2 = (a_2 - 1)b_2$. Then $a_2 = b_2 = 2$. Therefore, $a_1 = 2q$ and $b_1 = 2p$. Thus $(a, b) = (2pq, 2pq)$.

Case 2. $q \nmid b$. Then $\gcd(q, b) = 1$ and $q \mid a_1$. So we have $a_1 = qa_2$, for some positive integer a_2. Thus $pqa_2 = (a_2 - 1)b$. Since $\gcd(a_2, a_2 - 1) = 1$, we have two possible subcases.

Subcase 2.1 $pa_2 = b$ and $q = a_2 - 1$. Then $a_2 = q+1$ and so $a_1 = q(q+1)$. Thus $(a, b) = (pq(q+1), p(q+1))$.

Subcase 2.2 $a_2 = b$ and $pq = a_2 - 1$. Then $a_2 = pq+1$ and $a_1 = q(pq+1)$. Thus $(a, b) = (pq(pq + 1), pq + 1)$. □

When we divide (1.1) by m, it is easy to verify the following theorem, by Lemma 2.1.

Theorem 2.2. *For any positive integer m , the all positive integer solutions of (1.1) are*

$$(a,b) \in \{(\frac{p(p+q)}{m}, \frac{q(p+q)}{m}), (\frac{p(q+1)}{m}, \frac{pq(q+1)}{m}), (\frac{pq(p+1)}{m}, \frac{q(p+1)}{m}), (\frac{2pq}{m}, \frac{2pq}{m}), (\frac{pq(pq+1)}{m}, \frac{pq+1}{m})\} \cap \mathbb{Z} \times \mathbb{Z}.$$

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