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On the Positive Integer Solutions of $p^x + p^y = z^2$ in the Fibonacci and Lucas Numbers, where p is Prime

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Abstract

In 2019, Burshtein [1] showed that the equation $p^x+p^y = z^2$ has no positive integer solution, except for $p = 2, 3$. For $p = 2$, all solutions of the equation in the Fibonacci and Lucas numbers was investigated by Hashim [2]. In this paper, we prove that $(x, y, z) = (F_5, L_3, L_6)$ is the unique solution of the equation, when $p = 3$.

1 Introduction

Let p be prime and x, y, z be positive integers with $x \leq y$. In 2019, Burshtein [1] found that the Diophantine equation $p^x + p^y = z^2$ has no positive integer solution, except for $p = 2, 3$. In 2023, Hashim [2] studied all solutions of the equation in the Fibonacci and Lucas numbers, when $p = 2$. In this paper, we find all solutions of the equation in the Fibonacci and Lucas numbers, when $p = 3$. In other words, we solve the following equations:

$$3^{F_i} + 3^{F_j} = F_k^2, \tag{1.1}$$

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$$3^{F_i} + 3^{F_j} = L_k^2, \tag{1.2}$$

$$3^{L_i} + 3^{L_j} = F_k^2, \tag{1.3}$$

$$3^{L_i} + 3^{L_j} = L_k^2, \tag{1.4}$$

$$3^{F_i} + 3^{L_j} = F_k^2, \tag{1.5}$$

$$3^{F_i} + 3^{L_j} = L_k^2, \tag{1.6}$$

where the indices i, j, k are positive integers and F_n, L_n represent the nth terms of the Fibonacci and Lucas sequences, respectively, that are defined by the initial values $F_0 = 0, F_1 = 1$ and $L_0 = 2, L_1 = 1$ and the recurrence relations $F_n = F_{n-1} + F_{n-2}$ and $L_n = L_{n-1} + L_{n-2}$, where $n \geq 2$.

2 Preliminaries

Theorem 2.1. *[1] All positive integer solutions of the equation $3^x + 3^y = z^2$ are given by $(x, y, z) = (2n, 2n+1, 2 \cdot 3^n)$, where n is a positive integer.*

Proposition 2.2. *[3] Let n be a positive integer. Then*

1. $F_{n+1} + F_{n-1} = L_n$,
2. $F_{n+2} - F_{n-2} = L_n$.

Lemma 2.3. *If i is a positive integer with $i \geq 7$, then $F_i \geq L_j + 2$, for all positive integer $j \leq i - 2$.*

Proof. We prove this lemma by induction on i. It easy to see that $F_7 \geq L_j + 2$, for all positive integer $j \leq 5$. Suppose that $F_i \geq L_j + 2$, for all positive integer $j \leq i - 2$. Then $F_{i+1} \geq F_i \geq L_j + 2$, for all positive integer $j \leq i - 2$. It remains only to consider case $j = i - 1$. By Proposition 2.2 (2) and $i \geq 7$, we have $L_j + 2 = L_{i-1} + 2 = F_{i+1} - F_{i-3} + 2 \leq F_{i+1} - 1 < F_{i+1}$. □

3 Main Results

Theorem 3.1. *The equation $3^x + 3^y = z^2$ has only one positive integer solution in the Fibonacci and Lucas numbers; i.e., $(x, y, z) = (F_5, L_3, L_6)$.*

Proof. Consider (1.1) and (1.2). Without loss of generality, we may assume that $F_i \leq F_j$. By Theorem 2.1, we have $F_i = 2n$ and $F_j = 2n+1$, for some positive integer n. Then $i = 3$ and $j = 4$. From (1.1) and (1.2), we get $F_k = L_k = 6$, which is a contradiction. Next, we consider (1.3) and (1.4). Without loss of generality, we assume that $L_i \leq L_j$. By Theorem 2.1, we obtain $L_i = 2n$ and $L_j = 2n+1$, for some positive integer n. Then $L_j - L_i = 1$. Thus $i = 2$ and $j = 3$. This implies that $3 = 2n$, which is also a contradiction. Then (1.1)-(1.4) have no solution.

Finally, we consider (1.5) and (1.6). Assume that $i \leq j$. Then $F_i \leq F_j \leq L_j$. By Theorem 2.1, we obtain $F_i = 2n$ and $L_j = 2n+1$, for some positive integer n. Therefore, $L_j = F_i + 1$. By Proposition 2.2 (1) and $i \leq j$, we have $2F_{j-1} - 1 = F_i - F_j \leq 0$. Thus $j = 1, n = 0$, and so $i = 0$. This is impossible, since $i > 0$. Then $i > j$. For $i \geq 7$, we consider the following two cases:

Case 1. $F_i \leq L_j$. By Theorem 2.1, we obtain $F_i = 2n$ and $L_j = 2n+1$, for some positive integer n. Then $L_j = F_i + 1$. Thus, by Lemma 2.3, we have $j = i - 1$, and so $L_{i-1} = F_i + 1$. By Proposition 2.2 (1), we get $F_{i-2} = 1$. This is impossible, since $i \geq 7$.

Case 2. $F_i > L_j$. By Theorem 2.1, we obtain $F_i = 2n+1$ and $L_j = 2n$, for some positive integer n. Therefore $F_i = L_j + 1$. By Lemma 2.3, we see that $j = i-1$. Then $F_i = L_{i-1} + 1$. By Proposition 2.2 (1), we get $F_{i-2} = -1$, a contradiction.

Thus $i < 7$. Since $i > j$, we get $j < 7$. It easy to check that (1.5) has no solution and (1.6) has only one solution. That is $(x, y, z) = (F_5, L_3, L_6)$. □

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