International Journal of Mathematics and
Computer Science, **19**(2024), no. 2, 467–474

Distributive Lattices of Soft Ideals in Menger Algebras

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(Received October 1, 2023, November 6, 2023,
Published November 10, 2023)

Abstract

We introduce soft v-ideals, soft s-ideals and soft vs-ideals over a Menger algebra of rank n. Moreover, we investigate several properties of each of the types of soft ideals. Furthermore, we prove that the collection of all soft vs-ideals forms a distributive lattice.

1 Introduction

The problem of uncertainty or vagueness has been tackled for several years by philosophers, logicians, mathematicians and computer scientists. Many practical problems in engineering, economics, social science, environmental science, and medical science etc., cannot be resolved by classical methods due to these methods' inherent difficulties, such as the inadequacy of the theories of parameterization tools. The most successful approach to tackle this problem is through fuzzy set theory. A pair (U, m) is called a fuzzy set where U is a non-empty set and $m : U \longrightarrow [0, 1]$. Zadeh [10] introduced fuzzy set theory and described the control problems of complex systems using fuzzy systems.

Key words and phrases: Menger algebra, fuzzy set, soft set, soft Menger algebra, soft v-ideal, soft s-ideal, soft vs-ideal, distributive lattice.
AMS (MOS) Subject Classifications: 20N15.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

Let U be an universal set and let E be a set of parameters. Suppose A is a subset of E. A pair (F, A) is said to be a soft set over U if $F : A \longrightarrow P(U)$ is a mapping from A into the power set of U. Molodtsov [8] introduced the new concept of soft set as a completely generic mathematical tool for modeling uncertainties. As there is no limited condition to the description of objects, researchers can choose the form of parameters they need, which greatly simplifies the decision-making process and makes the process more efficient in the absence of partial information. Since the soft sets came into existence, some mathematicians started imposing and studying algebraic structures on soft sets. Because any soft set involves two component sets, namely, a universal set and a parameter set, interestingly, some researchers algebrized universal sets and others algebrized parameter sets, see [8]. Soft algebraic structures such as soft rings, soft semirings, soft semigroups, soft ternary semigroups, and soft algebras, have been widely studied, see [1, 2, 3, 4, 6, 7], and [9]. In this paper, we introduce soft v-ideals, soft s-ideals and soft vs-ideals over a Menger algebra of rank n. Several properties of each of the types of soft ideals are investigated, and it is proved that the collection of all soft vs-ideals forms a distributive lattice.

In [5], a non-empty set M together with an $(n+1)$-ary operation $S^{n,M} : M^{n+1} \to M$ satisfying the identity

$$\begin{aligned} & \bar{S}^{n,M}(x, \bar{S}^{n,M}(y_1, z_1, \ldots, z_n), \ldots, \bar{S}^{n,M}(y_n, z_1, \ldots, z_n)) \\ \approx\ & \bar{S}^{n,M}(\bar{S}^{n,M}(x, y_1, \ldots, y_n), z_1, \ldots, z_n) \end{aligned}$$

is called a *Menger algebra* of rank n. For example, let A be a non-empty set. The set of all n-ary operations $f : A^n \to A$ on A will be written as $O^n(A)$. Define the composition $S^{n,A} : (O^n(A))^{n+1} \to O^n(A)$ by

$$S^{n,A}(f, g_1, \ldots, g_n)(a_1, \ldots, a_n) = f(g_1(a_1, \ldots, a_n), \ldots, g_n(a_1, \ldots, a_n))$$

for all $f, g_1, \ldots, g_n \in O^n(A), a_1, \ldots, a_n \in A$. A pair $(O^n(A), S^{n,A})$ forms a Menger algebra of rank n. Consider the set $\mathbb{R}^+$ of all positive real numbers and the operation $\circ : (\mathbb{R}^+)^{n+1} \to \mathbb{R}^+$ defined by

$$\circ(x_0, x_1, \ldots, x_n) = x_0\sqrt{x_1 \ldots x_n}.$$

Then $(\mathbb{R}, \circ)$ forms a Menger algebra of rank n. Observe that a Menger algebra of rank 1 is a semigroup. Therefore, ideals in Menger algebras generalize the notion of ideals in semigroups. Let $(M, S^{n,M})$ be a Menger algebra of rank n. A non-empty subset A of M is called a *Menger subalgebra* of M if for any $x, y_1, \ldots, y_n \in A$, $S^{n,M}(x, y_1, \ldots, y_n) \in A$. A non-empty subset H of M is called an v-*ideal* (of M) if, for all $x, y_1, \ldots, y_n \in M$. If $y_1, \ldots, y_n \in H$, then $S^{n,M}(x, y_1, \ldots, y_n) \in H$. A non-empty subset H of M is called an s-*ideal* (of M) if, for all $y, x_1, \ldots, x_n \in M$. If $y \in H$, then $S^{n,M}(y, x_1, \ldots, x_n) \in H$. If H is both an v-ideal and an s-ideal of M, then H is called an vs-ideal of M. Let $(M_1, S_1^{n,M})$ be a Menger algebra of rank n. A mapping $\varphi : M \to M'$ is said to be a *homomorphism* if for any $x, y_1, \ldots, y_n \in M$, $\varphi(S^{n,M}(x, y_1, \ldots, y_n)) = S_1^{n,M}(\varphi(x), \varphi(y_1), \ldots, \varphi(y_n))$.

2 Main results

Let U be an universal set and let E be a set of parameters. A pair (F, E) is called a *soft set* (over U) if $F : E \to P(U)$ is a mapping from E into the power set $P(U)$ of the set U. Generally, let us consider a soft set as a pair (F, A) with $A \subseteq E$ and $F : A \to P(U)$. Let (U, E) be a soft set defined by $U : E \to P(U)$ such that $U(e) = U$ for all $e \in E$.

For example, a soft set (F, A) describes the attractiveness of the houses which Mr. X is going to buy. Let U be the set of houses under consideration. Let E = {expensive, beautiful, wooden, cheap, in the green surroundings, modern, being well repaired, being bad repaired} be the set of parameters; each parameter is a word or a sentence. Defining a soft set means pointing out expensive houses, beautiful houses, and so on. It is worth noting that the sets $F(a)$ may be arbitrary. Some of them may have non-empty intersection [8].

Definition 2.1. *Let (F, A) and (G, B) be soft sets over U. Then, (F, A) is a* soft subset *of (G, B) written as $(F, A)\tilde{\subseteq}(G, B)$, if $A \subseteq B$ and $F(a) \subseteq G(a)$ for all $a \in A$.*

Definition 2.2. *Let (F, A) and (G, B) be soft sets over U. Then (F, A) and (G, B) are said to be equal written as $(F, A) = (G, B)$ if $(F, A)\tilde{\subseteq}(G, B)$ and $(G, B)\tilde{\subseteq}(F, A)$.*

Hereafter, let $(M, S^{n,M})$ be a Menger algebra of rank n. For any subsets $A, B_1, \ldots, B_n$ of M, define

$$S^{n,M}(A, B_1, \ldots, B_n) = \{S^{n,M}(a, b_1 \ldots, b_n) \mid a \in A, b_1 \in B_1, \ldots, b_n \in B_n\}.$$

Definition 2.3. *Let $(F_i, A_i), 1 \leq i \leq n+1$ be soft sets over M. The* restricted product *(H, C) of (F_i, A_i) written by $(H, C) = (F_1, A_1)\widehat{\circ} \ldots \widehat{\circ}(F_n, A_n)$ if $C = \cap A_i$ and $H(x) = S^{n,M}(F_1(x), \ldots, F_{n+1}(x))$ for all $x \in C$.*

Definition 2.4. *A soft set* (F, A) *over* M *is called a* soft Menger algebra *(over* M*) if* $(F, A)\widehat{\circ}\ldots\widehat{\circ}(F, A)\tilde{\subseteq}(F, A)$.

Observe that a soft set (F, A) over M is a soft Menger algebra over M if and only if for any $a \in A$, $F(a) \neq \emptyset$ implies $F(a)$ is a Menger subalgebra of M. Indeed, we set $(H, C) = (F, A)\widehat{\circ}\ldots\widehat{\circ}(F, A)$. Then, $C = A$ and $H(a) = S^{n,M}(F(a), \ldots, F(a))$ for any $a \in A$. Assume (F, A) is a soft Menger algebra over M, then, $(H, C)\tilde{\subseteq}(F, A)$. Thus, $S^{n,M}(F(a) \ldots F(a)) \subseteq F(a)$ for any $a \in A$. Therefore, for any $a \in A$, $F(a) \neq \emptyset$ implies $F(a)$ is a Menger subalgebra of M. Conversely, suppose for any $a \in A$, $F(a) \neq \emptyset$ implies $F(a)$ is a Menger subalgebra of M. We have to show that $(H, C)\tilde{\subseteq}(F, A)$. Let $a \in A$. Clearly, if $F(a) = \emptyset$, then $H(a) \subseteq F(a)$. If $F(a) \neq \emptyset$, then $H(a) = S^{n,M}(F(a), \ldots, F(a)) \subseteq F(a)$. Therefore, $(H, C)\tilde{\subseteq}(F, A)$, and (F, A) is a soft Menger algebra over M.

Definition 2.5. *A soft set* (F, A) *over* M *is called a* soft v-ideal *(over* M*) if* $(M, E)\widehat{\circ}(F, A)\widehat{\circ}\ldots\widehat{\circ}(F, A)\tilde{\subseteq}(F, A)$.

Proposition 2.6. *A soft set* (F, A) *over* M *is a soft* v*-ideal over* M *if and only if, for any* $a \in A$, $F(a) \neq \emptyset$ *implies* $F(a)$ *is an* v*-ideal of* M.

Proof. Assume (F, A) is a soft v-ideal over M; then, $(M, E)\widehat{\circ}(F, A)\widehat{\circ}\ldots\widehat{\circ}(F, A)$ $\tilde{\subseteq}(F, A)$. Let $a \in A$ such that $F(a) \neq \emptyset$. By assumption, $S^{n,M}(M, F(a), \ldots, F(a))$ $\subseteq F(a)$. Therefore, $F(a)$ is an v-ideal of M. Conversely, assume, for any $a \in A$, $F(a) \neq \emptyset$ implies $F(a)$ is an v-ideal of M. Setting $(H, C) = (M, E)\widehat{\circ}(F, A)\widehat{\circ}\ldots\widehat{\circ}(F, A)$; we have to show that $(H, C)\tilde{\subseteq}(F, A)$. Obviously, $C \subseteq A$. Let $a \in A$. If $F(a) = \emptyset$, then $H(a) \subseteq F(a)$. If $F(a) \neq \emptyset$, then by assumption, $H(a) = S^{n,M}(M, F(a), \ldots, F(a)) \subseteq F(a)$. Hence, (F, A) is a soft v-ideal over M. □

Definition 2.7. *A soft set* (F, A) *over* M *is called a* soft s-ideal *(over* M*) if* $(F, A)\widehat{\circ}(M, E)\widehat{\circ}\ldots(M, E)\tilde{\subseteq}(F, A)$.

Proposition 2.8. *A soft set* (F, A) *over* M *is a soft* s*-ideal if and only if, for any* $a \in A$, $F(a) \neq \emptyset$ *implies* $F(a)$ *is an* s*-ideal of* M.

Proof. Assume (F, A) is a soft s-ideal over M; then $(F, A)\widehat{\circ}(M, E)\widehat{\circ}\ldots\widehat{\circ}(M, E)$ $\tilde{\subseteq}(F, A)$. Let $a \in A$ such that $F(a) \neq \emptyset$. By assumption, $S^{n,M}(F(a), M, \ldots, M)$ $\subseteq F(a)$. Therefore, $F(a)$ is an s-ideal of M. Conversely, assume for any $a \in A$, $F(a) \neq \emptyset$ implies $F(a)$ is an s-ideal of M. Setting $(H, C) = (F, A)\widehat{\circ}(M, E)\widehat{\circ}\ldots\widehat{\circ}(M, E)$; we have to show that $(H, C)\tilde{\subseteq}(F, A)$. Clearly, $C = A$. Let $a \in A$. If $F(a) = \emptyset$, then $H(a) \subseteq F(a)$. If $F(a) \neq \emptyset$, then by assumption $H(a) = S^{n,M}(F(a), M, \ldots, M) \subseteq F(a)$. Hence, (F, A) is a soft s-ideal over M. □

Definition 2.9. *A soft set (F, A) over M is called a* soft vs-ideal *(over M) if (F, A) is both a soft v-ideal and a soft s-ideal over M.*

Proposition 2.10. *A soft set (F, A) over M is a soft vs-ideal if and only if, for any $a \in A$, $F(a) \neq \emptyset$ implies $F(a)$ is an vs-ideal of M.*

Proof. From Proposition 2.6 and Proposition 2.8, the assertion follows. □

For example, consider a Menger algebra $(M, S^{1,M})$ of rank 1 such that $M = \{1, a, b, c, d\}$ and

$S^{1,M}$	1	a	b	c	d
1	1	a	b	c	d
a	a	a	a	a	a
b	b	a	a	a	a
c	c	a	a	b	a
d	d	a	a	b	b

Define $F : M \to P(M)$ by $F(x) = \{y \in M \mid \exists u \in M, y = S^{1,M}(x, u)\}$. Here, M is considered as a set of parameters. We have $F(a) = \{a\}, F(b) = \{a, b\}, F(c) = \{a, b, c\}, F(d) = \{a, b, d\}$. Since $F(x)$ is a Menger subalgebra of M for all $x \in M$, (F, M) is a soft Menger algebra over M. Observe that (F, M) is a soft s-ideal over M.

Definition 2.11. *Let (F, A) and (G, B) be soft sets over U.*

(1) The basic intersection of (F, A) and (G, B) is defined to be the soft set $(H, C) = (F, A) \wedge (G, B)$ where $C = A \times B$ and $H(x, y) = F(x) \cap G(y)$ for all $(x, y) \in C$.

(2) The intersection of (F, A) and (G, B) is defined to be the soft set $(H, C) = (F, A) \cap (G, B)$ where $C = A \cap B$ and $H(x) = F(x) \cap G(x)$ for all $x \in C$.

(3) The basic union of (F, A) and (G, B) is defined to be the soft set $(H, C) = (F, A) \vee (G, B)$ where $C = A \times B$ and $H(x, y) = F(x) \cup G(y)$ for all $(x, y) \in C$.

(4) The union of (F, A) and (G, B) is defined to be the soft set $(H, C) = (F, A) \cup (G, B)$ where $C = A \cup B$ and

$$H(x) = \begin{cases} F(x) & , \text{if } x \in A \setminus B; \\ G(x) & , \text{if } x \in B \setminus A; \\ F(x) \cup G(x) & , \text{if } x \in A \cap B. \end{cases}$$

Proposition 2.12. *Let (F, A) and (G, B) be soft Menger algebras over M such that $A \cap B \neq \emptyset$. Then $(F, A) \cap (G, B)$ is a soft Menger algebra over M.*

Proof. Setting $(H, C) = (F, A) \cap (G, B)$; then $C = A \cap B$ and $H(x) = F(x) \cap G(x)$ for all $x \in C$. Thus, for $x \in C$, $F(x) \cap G(x) \neq \emptyset$ implies $F(x) \cap G(x)$ is a Menger subalgebra of M. Therefore, $(F, A) \cap (G, B)$ is a soft Menger algebra over M. □

Similarly, we have the following:

Proposition 2.13. *Let (F, A) and (G, B) be soft Menger algebras over M such that $A \cap B = \emptyset$. Then $(F, A) \cup (G, B)$ is a soft Menger algebra over M.*

Proposition 2.14. *Let (F, A) and (G, B) be soft Menger algebras over M. Then $(F, A) \wedge (G, B)$ is a soft Menger algebra over M.*

Proposition 2.15. *Let (F, A) and (G, B) be soft vs-ideals over M such that $A \cap B \neq \emptyset$. Then $(F, A) \cap (G, B)$ is a soft vs-ideal over M contained in both (F, A) and (G, B).*

Proof. Setting $(H, C) = (F, A) \cap (G, B)$; then $C = A \cap B$ and $H(x) = F(x) \cap G(x)$ for all $x \in C$. Let $x \in C$. From $F(x)$ and $G(x)$ are vs-ideals of M for all $x \in C$, it follows that, for all $x \in C$, $F(x) \cap G(x) = \emptyset$ or $F(x) \cap G(x)$ is an vs-ideal of M. Hence $(F, A) \cap (G, B)$ is a soft vs-ideal over M. We have $C \subseteq A$, $C \subseteq B$, $H(x) \subseteq F(x)$ for all $x \in C$ and $H(x) \subseteq G(x)$ for all $x \in C$. Hence, $(H, C) \tilde{\subseteq} (F, A)$ and $(H, C) \tilde{\subseteq} (G, B)$. □

Similarly we have the following:

Proposition 2.16. *Let (F, A) and (G, B) be soft vs-ideals over M such that $A \cap B \neq \emptyset$. Then, $(F, A) \cup (G, B)$ is a soft vs-ideal over M containing both (F, A) and (G, B).*

Proposition 2.17. *If (F, A) and (G, B) are soft vs-ideals over M, then $(F, A) \wedge (G, B)$ is a soft vs-ideal over M.*

Proposition 2.18. *If (F, A) and (G, B) are soft vs-ideals over M, then $(F, A) \vee (G, B)$ is a soft vs-ideal over M.*

Definition 2.19. *Let (F, A) and (G, B) be soft Menger algebras over M such that $(F, A) \tilde{\subseteq} (G, B)$. Then (F, A) is called a* Menger subalgebra *(resp.,* soft v-ideal, soft s-ideal, soft vs-ideal*) of (G, B) if for all $a \in A$, $F(a)$ is a Menger subalgebra (resp., an v-*ideal*, an s-*ideal*, an vs-*ideal*) of $G(a)$.*

Let $(M_1, S_1^{n,M})$ be a Menger algebra of rank n. Let $\varphi : M \to M_1$ be a homomorphism. For a soft set (F, A) over M, define a soft set $(\varphi(F), A)$ over M_1 where $\varphi(F) : A \to P(M_1)$ such that $\varphi(F)(a) = \varphi(F(a))$ for all $a \in A$.

Theorem 2.20. *If (F, A) and (G, B) are soft Menger algebras over M such that (G, B) is a soft Menger subalgebra of (F, A), then $(\varphi(F), A)$ and $(\varphi(G), B)$ are soft Menger algebras over M_1 and $(\varphi(G), B)$ is a soft Menger subalgebra of $(\varphi(F), A)$.*

Proof. Since φ is a homomorphism, $\varphi(F(a))$ is a Menger subalgebra of M_1 for all $a \in A$. So, $(\varphi(F), A)$ is a soft Menger algebra over M_1. Similarly, $(\varphi(G), B)$ is a soft Menger algebra over M_1. Let $b \in B$. Since $G(b)$ is a Menger subalgebra of $F(b)$, $\varphi(G(b))$ is a Menger subalgebra of $\varphi(F(b))$. Therefore, $(\varphi(G), B)$ is a soft Menger subalgebra of $(\varphi(F), A)$. □

Similarly, we have the following:

Theorem 2.21. *If (F, A) and (G, B) are soft Menger algebras over M such that (G, B) is a soft vs-ideal of (F, A), then $(\varphi(G), B)$ is a soft vs-ideal of $(\varphi(F), A)$.*

Let the parameter set A be fixed. We denote the collection of all the soft vs-ideals with parameter set A by $\mathcal{S}I(M)_A$. Then,

Theorem 2.22. *$\mathcal{S}I(M)_A$ forms a distributive lattice.*

Proof. From Propositions 2.15 and Proposition 2.16, it follows that $\mathcal{S}I(M)_A$ forms a lattice. We have to show that the lattice is distributive. Let (F, A), (G, A), $(H, A) \in \mathcal{S}I(M)_A$. Setting

$$(K, A) \cap (H, A) = ((F, A) \cup (G, A)) \cap (H, A)$$

where $(K, A) = (F, A) \cup (G, A)$ and, for all $a \in A$, $K(a) = F(a) \cup G(a)$. Setting

$$(M, A) = (K, A) \cap (H, A)$$

where, for all $a \in A$,

$$M(a) = K(a) \cap H(a) = (F(a) \cup G(a)) \cap H(a) = (F(a) \cap H(a)) \cup (G(a) \cap H(a)).$$

Setting

$$(N, A) = (F, A) \cap (H, A)$$

where, for all $a \in A$, $N(a) = F(a) \cap H(a)$.
Setting

$$(P, A) = (G, A) \cap (H, A)$$

where, for all $a \in A$, $P(a) = G(a) \cap H(a)$. We have $M(a) = N(a) \cup P(a)$ for all $a \in A$. Therefore,

$$((F, A) \cup (G, A)) \cap (H, A) = ((F, A) \cap (H, A)) \cup ((G, A) \cap (H, A)).$$

Hence $\mathcal{S}I(M)_A$ forms a distributive lattice. □

Acknowledgment. The Research on "Distributive lattices of soft ideals in Menger algebras" by Khon Kaen University has received funding support from the National Science, Research and Innovation Fund (NSRF).

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