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Prime one-sided ideals in ternary semirings

Panuwat Luangchaisri, Thawhat Changphas

Department of Mathematics
Faculty of Science
Khon Kaen University
Khon Kaen 40002, Thailand

email: panulu@kku.ac.th, thacha@kku.ac.th

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Abstract

In this paper, we study right weakly regular ternary semirings and fully prime right ternary semirings. Let T be a ternary semiring with absorbing zero and identity. We prove the following:
(1) T is right weakly regular if and only if $[AAA] = A$ for each right ideal A of T, and
(2) T is a fully prime right ternary semiring if and only if T is right weakly regular and for ideals A, B and C of T one of the following assertions holds: $A \subseteq B \cap C$; $B \subseteq A \cap C$; $C \subseteq A \cap B$.

1 Introduction

Algebraic structures of ternary semirings: regularities, radicals, ideals, spectrums, special elements, have been extensively studied by many authors [1], [2], [3], [6], [7], [8], [9], [10], [11]. Prime one-sided ideals in semirings and in Γ-semirings have been introduced and studied in [5] and [12]. In this paper, we consider right weakly regular ternary semirings and fully prime right ternary semirings. Let T be a ternary semiring with absorbing zero and identity. We prove that T is right weakly regular if and only if $[AAA] = A$ for each

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right ideal A of T (let $[\ \ ]$ denote the ternary operation on T). Moreover, a characterization of fully prime right ternary semirings is given in terms of right weakly regularity and two-sided ideals. Indeed, it is proved that T is a fully prime right ternary semiring if and only if T is right weakly regular and for two-sided ideals A, B and C of T one of the following assertions holds: $A \subseteq B \cap C$; $B \subseteq A \cap C$; $C \subseteq A \cap B$.

Following [4] and [11], a *semigroup* $(S, *)$ consists of a nonempty set S together with a binary operation $*$ on S satisfying the associative law: $a * (b * c) = (a * b) * c$ for all $a, b, c \in S$. A semigroup S is said to be *commutative* if $a * b = b * a$ for all $a, b \in S$. A *ternary semiring* $(T, +, [\ \ ])$ consists of a nonempty set T which the binary operations of addition $(+)$ and ternary multiplication $([\ \ ])$ have been defined such that the following assertions are satisfied:

(1) $(T, +)$ is a commutative semigroup;

(2) $(T, [\ \ ])$ is a ternary semigroup; i.e.,

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]]$$

for all $x_1, \ldots, x_5 \in T$;

(3) multiplication distributes over addition from either side; i.e.,

- $[(a + b)cd] = [acd] + [bcd]$;
- $[a(b + c)d] = [abd] + [acd]$;
- $[ab(c + d)] = [abc] + [abd]$,

for all $a, b, c, d \in T$.

For $A, B, C, D \subseteq T$, define the set product $[ABC]$ to be the set of finite sums of the form $\Sigma_{finite}[a_ib_ic_i]$, where $a_i \in A, b_i \in B, c_i \in C$, i.e.,

$$[ABC] = \{\Sigma_{finite}[a_ib_ic_i] \mid a_i \in A, b_i \in B, c_i \in C\}.$$

It is observed that for $A_1, A_2, B_1, B_2, C_1, C_2 \subseteq T$, if $A_1 \subseteq A_2, B_1 \subseteq B_2$, and $C_1 \subseteq C_2$, then $[A_1B_1C_1] \subseteq [A_2B_2C_2]$. An element 0 of a ternary semiring T is called an *absorbing zero* if $0 + a = a + 0 = a$ for all $a \in T$ and $[ab0] = [a0b] = [0ab] = 0$ for all $a, b \in T$. An element 1 of a ternary semiring T is called an *identity* of T if $[a11] = [1a1] = [11a] = a$, for all $a \in T$. *Hereafter, we deal with a ternary semiring T with absorbing zero* 0 *and identity* 1. A nonempty subset A of a ternary semiring T is called a *right ideal* (of T) if $a+b \in A$, for all $a, b \in A$, and $[axy] \in A$ for all $x, y \in T$ and $a \in A$; i.e., $[ATT] \subseteq A$. For left ideals can be defined similarly. A nonempty subset A of a ternary semiring T is called a *middle ideal* (of T) if $a+b \in A$ for all $a, b \in A$, and $[xay] \in A$ for all $x, y \in T$ and $a \in A$; i.e., $[TAT] \subseteq A$. If A is a left, a middle and a right ideal of T, then A is called an *ideal* (of T). Let a be an element of a ternary semiring T. The *principal right ideal* of T generated by a is of the form $[\{a\}TT]$ (it is abbreviated by $[aTT]$); the *principal left ideal* of T generated by a is of the form $[TTa]$; the *principal middle ideal* of T generated by a is of the form $[TaT]$.

2 Main results

Let us recall that T is a ternary semiring with absorbing zero 0 and identity 1. We begin this section with the definition of prime right ideal of T.

Definition 2.1. *Let P be a right ideal of T. Then P is said to be* prime *if for any right ideal A_1, A_2 and A_3 of T, $[A_1A_2A_3] \subseteq P$ implies $A_i \subseteq P$, for some $1 \leq i \leq 3$.*

Theorem 2.2. *Let P be a right ideal of T. Then P is prime if and only if for any $a_1, a_2, a_3 \in T$, $[a_1TTa_2TTa_3] \subseteq P$ implies $a_i \in P$ for some $1 \leq i \leq 3$.*

Proof. Assume that P is prime. Let $a_1, a_2, a_3 \in T$ be such that $[a_1TTa_2TTa_3] \subseteq P$. Consider

$$[[a_1TT][a_2TT][a_3TT]] = [[[a_1TT][a_2TT]a_3]TT] \subseteq [PTT] \subseteq P.$$

Since $[a_1TT], [a_2TT]$, and $[a_3TT]$ are right ideals of T, $[a_iTT] \subseteq P$, for some $1 \leq i \leq 3$. Hence $a_i \in P$, for some $1 \leq i \leq 3$. Conversely, assume that for any $a_1, a_2, a_3 \in T$, $[a_1TTa_2TTa_3] \subseteq P$ implies $a_i \in P$ for some $1 \leq i \leq 3$. Let A_1, A_2 and A_3 be right ideals of T such that $[A_1A_2A_3] \subseteq P$. Suppose that there exist $a_1 \in A_1 \setminus P$ and $a_2 \in A_2 \setminus P$. Let $a_3 \in A_3$. Then

$$[a_1TTa_2TTa_3] \subseteq [A_1TTA_2TTA_3] \subseteq [A_1A_2A_3] \subseteq P.$$

By assumption, $a_i \in P$, for some $1 \leq i \leq 3$. Thus $a_3 \in P$. Therefore, $A_3 \subseteq P$ and hence P is prime. □

Definition 2.3. *Let P be a right ideal of T. Then P is said to be* semiprime *if for any right ideal A of T, $[AAA] \subseteq P$ implies $A \subseteq P$.*

Observe that every prime right ideal is semiprime.

Theorem 2.4. *Let P be a right ideal of T. Then P is semiprime if and only if for any $a \in T$, $[aTTaTTa] \subseteq P$ implies $a \in P$.*

Proof. Assume that P is semiprime. Let $a \in T$ be such that $[aTTaTTa] \subseteq P$. Consider

$$[[aTT][aTT][aTT]] = [[[aTT][aTT]a]TT] \subseteq [PTT] \subseteq P.$$

Since $[aTT]$ is a right ideal of T, $[aTT] \subseteq P$. Hence $a \in P$. Conversely, assume that for any $a \in T$, $[aTTaTTa] \subseteq P$ implies $a \in P$. Let A be a right ideal of T such that $[AAA] \subseteq P$. Let $a \in A$. Then

$$[aTTaTTa] \subseteq [ATTATTA] \subseteq [AAA] \subseteq P.$$

By assumption, $a \in P$. Therefore, $A \subseteq P$ and hence P is semiprime. $\square$

Definition 2.5. *Let A be a right ideal of T. Then A is said to be* irreducible *if for any right ideal B_1, B_2 and B_3 of T, $B_1 \cap B_2 \cap B_3 = A$ implies $B_i = A$, for some $1 \leq i \leq 3$.*

Definition 2.6. *Let A be a right ideal of T. Then A is said to be* strongly irreducible *if for any right ideal B_1, B_2 and B_3 of T, $B_1 \cap B_2 \cap B_3 \subseteq A$ implies $B_i \subseteq A$ for some $1 \leq i \leq 3$.*

Definition 2.7. *A proper right ideal A of T is said to be* maximal *if there is no any proper right ideal of T containing A properly.*

Theorem 2.8. *Let A be a right ideal of T. If $x \notin A$, then there exists an irreducible right ideal containing A and not containing x.*

Proof. Assume that $x \notin A$. Clearly, the set of right ideals of T containing A and not containing x is nonempty. Consider a set $\{A_i \mid i \in \Lambda\}$ of a chain of right ideals of T containing A and not containing x. Then $\cup_{i \in \Lambda} A_i$ is a right ideals of T containing A and not containing x. By Zorn's lemma, the set of right ideals of T containing A and not containing x contains a maximal element, denoted M. To show that M is irreducible, let B_1, B_2 and B_3 be right ideals of T such that $B_1 \cap B_2 \cap B_3 = M$. Suppose that $M \subset B_1$, $M \subset B_2$ and $M \subset B_3$. Then $x \in B_1$, $x \in B_2$ and $x \in B_3$. Since $x \notin M$, $x \notin B_1$, $x \notin B_2$ or $x \notin B_3$. This is a contradiction. Hence $M = B_i$, for some $1 \leq i \leq 3$. $\square$

Theorem 2.9. *Any proper right ideal A of T is the intersection of irreducible right ideals of T containing A.*

Proof. Let A be a proper right ideal of T, $\{A_i \mid i \in \Lambda\}$ the set of irreducible right ideals of T containing A. Then $A \subseteq \cap A_i$. If $x \notin A$, then there exists an irreducible right ideal B of R such that $A \subseteq B$ and $x \notin B$. Then $x \notin \cap A_i$. Hence $\cap A_i \subseteq A$. Consequently, $A = \cap A_i$. □

Theorem 2.10. *Let P be a right ideal of T. If P is strongly irreducible semiprime, then P is prime.*

Proof. Assume that P is strongly irreducible semiprime. To show that P is prime, let A_1, A_2 and A_3 be right ideals of T such that $[A_1A_2A_3] \subseteq P$. We have

$$(A_1 \cap A_2 \cap A_3)(A_1 \cap A_2 \cap A_3)(A_1 \cap A_2 \cap A_3) \subseteq [A_1A_2A_3] \subseteq P.$$

Since $A_1 \cap A_2 \cap A_3$ is a right ideal of T and P is semiprime, $A_1 \cap A_2 \cap A_3 \subseteq P$. Since P is strongly irreducible, it follows that $A_i \subseteq P$, for some $1 \leq i \leq 3$. Hence P is prime. □

Definition 2.11. *A ternary semiring T is said to be* right weakly regular *if $a \in [[aTT][aTT][aTT]]$ for all $a \in T$.*

Theorem 2.12. *The following assertions are equivalent:*

(1) T is right weakly regular;

(2) $[AAA] = A$, for each right ideal A of T.

Proof. Assume that T is right weakly regular. Let A be a right ideal of T. Then $[AAA] \subseteq [ATT] \subseteq A$. If $a \in A$, then, by assumption,

$$a \in [(aTT)(aTT)(aTT)] \subseteq [(ATT)(ATT)(ATT)] \subseteq [AAA].$$

Then $A \subseteq [AAA]$. Hence $A = [AAA]$. Conversely, assume that $[AAA] = A$, for each right ideal A of T. To show that T is right weakly regular, let $a \in T$. Since $[aTT]$ is a right ideal of T, $[[aTT][aTT][aTT]] = [aTT]$, $a \in [aTT] = [[aTT][aTT][aTT]]$. Therefore, T is right weakly regular.

□

Theorem 2.13. *T is right weakly regular if and only if every right ideal of T is semiprime.*

Proof. Assume that T is right weakly regular. Let A be a right ideal of T. Let B be a right ideal of T such that $[BBB] \subseteq A$. By assumption and Theorem 2.12, $B = [BBB]$. Thus $B \subseteq A$. Hence A is semiprime. Conversely, assume that every right ideal of T is semiprime. To show that T is right weakly regular, let C be a right ideal of T. Since $[CCC]$ is a right ideal of T, $[CCC]$ is semiprime. Since $[CCC] \subseteq [CCC]$, it follows that $C \subseteq [CCC]$. Since $[CCC] \subseteq C \subseteq [CCC]$, $[CCC] = C$. By Theorem 2.12, T is right weakly regular. □

Definition 2.14. *A ternary semiring T is called a* fully prime right ternary semiring *if all right ideals of T are prime right ideals. For a* fully semiprime right ternary semiring *can be defined similarly.*

Theorem 2.15. *If T is a fully prime right ternary semiring, then T is right weakly regular and for any ideal A, B and C, $A \subseteq B \cap C$ or $B \subseteq A \cap C$ or $C \subseteq A \cap B$.*

Proof. If T is a fully prime right ternary semiring, then all right ideals of T are prime right ideals of T. Since every prime right ideal is semiprime and Theorem 2.13, T is right weakly regular. Let A, B and C be ideals of T. Then $A \cap B \cap C$ is a right ideal of T. By assumption, $A \cap B \cap C$ is prime. Since $[ABC] \subseteq A \cap B \cap C$, $A \subseteq A \cap B \cap C$ or $B \subseteq A \cap B \cap C$ or $C \subseteq A \cap B \cap C$. This means that $A = A \cap B \cap C$ or $B = A \cap B \cap C$ or $C = A \cap B \cap C$. Therefore, $A \subseteq B \cap C$ or $B \subseteq A \cap C$ or $C \subseteq A \cap B$. □

Theorem 2.16. *If T is right weakly regular and for any ideal A, B and C, $A \subseteq B \cap C$ or $B \subseteq A \cap C$ or $C \subseteq A \cap B$, then T is a fully prime right ternary semiring.*

Proof. Assume that T is right weakly regular and for any ideal A, B and C, $A \subseteq B \cap C$ or $B \subseteq A \cap C$ or $C \subseteq A \cap B$. We show that T is a fully prime right ternary semiring. Let P be a right ideal of T. To show that P is prime, let D_1, D_2 and D_3 be right ideals of T such that $[D_1 D_2 D_3] \subseteq P$. We have $D_1 \subseteq D_2 \cap D_3$ or $D_2 \subseteq D_1 \cap D_3$ or $D_3 \subseteq D_1 \cap D_2$; $[D_1 D_1 D_1] = D_1$, $[D_2 D_2 D_2] = D_2$ and $[D_3 D_3 D_3] = D_3$. If $D_1 \subseteq D_2 \cap D_3$, then $D_1 = [D_1 D_1 D_1] \subseteq [D_1 D_2 D_3] \subseteq P$. Similarly, for $D_2 \subseteq D_1 \cap D_3$ or $D_3 \subseteq D_1 \cap D_2$. Hence P is prime. □

Now we give a characterization of a fully prime right ternary semiring followed by Theorems 2.15 and 2.16.

Theorem 2.17. *Let T be a ternary semiring. Then T is a fully prime right ternary semiring if and only if T is right weakly regular and for any ideal A, B and C, $A \subseteq B \cap C$ or $B \subseteq A \cap C$ or $C \subseteq A \cap B$.*

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