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On the Diophantine Equations $(p+a)^x - p^y = z^2$ and $p^x - (p+a)^y = z^2$

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Abstract

In this article, we show that $(x, y, z) = (0, 0, 0)$ is the unique non-negative integer solution of the Diophantine equations $(p+a)^x - p^y = z^2$ and $p^x - (p+a)^y = z^2$, where a is a positive integer and p is prime with some conditions.

1 Introduction

In recent years, all non-negative integer solutions of the Diophantine equation $a^x - b^y = z^2$ has been extensively investigated, where a and b are fixed positive integers. Some of these can be seen in [4], [6], [12], [16], [17], [18], [19] and [20]. Moreover, many researchers extended solving the equation for a or b being prime. In 2019, Burshtein [3] found all positive integer solutions of the equations $(p+1)^x - p^y = z^2$ and $p^x - (p+1)^y = z^2$, where p is prime and $x + y = 2, 3, 4$. In the same year, Burshtein [5] also solved the equation $p^x - p^y = z^2$, when p is prime.

Additionally, in 2020, Elshahed and Kamarulhaili [7] studied the non-negative integer solutions of the equation $(4^n)^x - p^y = z^2$, where p is odd

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prime and n is a positive integer. Buosi el at. ([1], [2]) found the non-negative integer solutions for the equation $p^x - 2^y = z^2$, where $p = k^2 + n$ is prime, k is a positive integer and $n \in \{2, 4\}$. In 2022, Orosram and Unchai [9] solved the equation $2^{2nx} - p^y = z^2$, where n is a positive integer and p is prime. Tadee [11] showed that $(x, y, z) = (0, 0, 0)$ is the unique non-negative integer solution of the equation $(p+6)^x - p^y = z^2$, where p is prime with $p \equiv 1 \pmod{28}$. In 2023, Tadee [13] also studied the equation $3^x - p^y = z^2$, where p is prime. Moreover, Tadee and Laomalaw [14] proved that $(x, y, z) = (0, 0, 0)$ is the unique non-negative integer solution of the equation $(p+2)^x - p^y = z^2$, where p is prime with $p \equiv 5 \pmod{24}$.

Motivated by the above papers, we give some conditions such that $(x, y, z) = (0, 0, 0)$ is the unique non-negative integer solution of two equations:

$$(p + a)^x - p^y = z^2 \tag{1.1}$$

and

$$p^x - (p + a)^y = z^2, \tag{1.2}$$

where a is a positive integer and p is prime.

2 Preliminaries

In this section, we begin by introducing an important and useful theorem, which was proved by Mihăilescu [8] in 2004:

Theorem 2.1. *[8] (Mihăilescu's Theorem) The equation* $a^x - b^y = 1$ *has the unique solution* $(a, b, x, y) = (3, 2, 2, 3)$*, where* a, b, x *and* y *are positive integers with* $\min\{a, b, x, y\} > 1$.

Next, we recall the basic concepts of order of an integer, primitive root and Legendre symbol (See [10]).

Definition 2.2. *Let* m *be a positive integer. Then the Euler phi function, denoted by* $\varphi(m)$*, is the cardinality of the set* $\{1 < n < m : \gcd(n, m) = 1\}$.

Definition 2.3. *Let* m *be a positive integer and let* a *be any integer relatively prime to* m*. If* h *is the least positive integer such that* $a^h \equiv 1 \pmod{m}$*, then* h *is called the order of* a *modulo* m *and is denoted by* $ord_m a = h$.

Definition 2.4. *Let m be a positive integer and let a be any integer relatively prime to m. If $ord_m a = \varphi(m)$, then a is called a primitive root modulo m.*

Theorem 2.5. *Let j, k be positive integers and $ord_m a = h$. Then $a^j \equiv a^k \pmod{m}$ if and only if $j \equiv k \pmod{h}$.*

Theorem 2.6. *Let a be a positive integer and let p be prime with $\gcd(a,p) = 1$. If $ord_p a = p-1$, then $a^{\frac{p-1}{2}} \equiv -1 \pmod{p}$.*

Definition 2.7. *Let a be a positive integer and let p be odd prime. The Legendre symbol, $\left(\frac{a}{p}\right)$, is defined by*

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } x^2 \equiv a \pmod{p} \text{ is solvable} \\ 0 & \text{if } p \mid a \\ -1 & \text{if } x^2 \equiv a \pmod{p} \text{ is not solvable} \end{cases}.$$

Theorem 2.8. *[15] Let p and q be distinct odd prime with $q \equiv 1 \pmod{4}$. Then*

$$\left(\frac{q}{p}\right) = \begin{cases} 1 & \text{if } p \equiv q + r^{S_1}q + r^{S_1} \pmod{2q} \\ -1 & \text{if } p \equiv q + r^{S_2}q + r^{S_2} \pmod{2q} \end{cases},$$

where $S_1 \in \{2, 4, 6, ..., q-1\}$, $S_2 \in \{1, 3, 5, ..., q-2\}$ and r is a primitive root modulo q.

3 Main Results

Theorem 3.1. *Let a be a positive integer with $a \equiv 2 \pmod{4}$ and let p be prime with $p \equiv 1 \pmod{4}$. If $a \equiv -1, 1 \pmod{p}$, then the Diophantine equation (1.1) has the unique non-negative integer solution; i.e., $(x, y, z) = (0, 0, 0)$.*

Proof. Let x, y and z be non-negative integers such that the equation (1.1) is true. We consider the following four cases:
Case 1. $x = 0$ and $y = 0$. From (1.1), we get $z = 0$. Then $(x, y, z) = (0, 0, 0)$.
Case 2. $x = 0$ and $y \geq 1$. From (1.1), it follows that $1 - p^y = z^2$ and so $z^2 < 0$, a contradiction.
Case 3. $x \geq 1$ and $y = 0$. From (1.1), we have $(p+a)^x - z^2 = 1$. Assume that $x = 1$. Then $p + a - 1 = z^2$. Since $a \equiv 2 \pmod{4}$ and $p \equiv 1 \pmod{4}$, we get $z^2 \equiv 2 \pmod{4}$, which is impossible since $z^2 \equiv 0, 1 \pmod{4}$. Thus $x > 1$. If $z = 0$, then $(p+a)^x = 1$ and so $x = 0$, a contradiction. If $z = 1$,

then $(p+a)^x = 2$. Therefore, $x = 1$ and $p + a = 2$, a contradiction. Thus $z > 1$, which contradicts Theorem 2.1.
Case 4. $x \geq 1$ and $y \geq 1$. From (1.1) and $z^2 \equiv 0, 1 \pmod 4$, we get $(-1)^x - 1 \equiv 0, 1 \pmod 4$. Thus $x = 2k$ for some positive integer k. From (1.1), we have $[(p+a)^k - z][(p+a)^k + z] = p^y$. Since p is prime, there exists a non-negative integer u such that $(p+a)^k - z = p^u$ and $(p+a)^k + z = p^{y-u}$. Thus $2(p+a)^k = p^u[p^{y-2u}+1]$. Assume that $u > 0$. Since p is prime, we have $p \mid a$, which is impossible since $a \equiv -1, 1 \pmod p$. Then $u = 0$. It follows that $2(p+a)^k = p^y + 1$ and so $2a^k \equiv 1 \pmod p$, which is impossible since $a \equiv -1, 1 \pmod p$ and $p \neq 3$. □

By Theorem 3.1, if $a = p + 1$, then we have the following corollary:

Corollary 3.2. *If p is prime with $p \equiv 1 \pmod 4$, then the Diophantine equation $(2p+1)^x - p^y = z^2$ has the unique solution $(x, y, z) = (0, 0, 0)$, where x, y and z are non-negative integers.*

Theorem 3.3. *Let a and p be distinct odd prime with $a \equiv 1 \pmod{4p}$ and $p \equiv a + r^{S_2} a + r^{S_2} \pmod{2a}$, where $S_2 \in \{1, 3, 5, ..., a-2\}$ and r is a primitive root modulo a. If $x \neq 1$, then the Diophantine equation (1.1) has the unique non-negative integer solution; i.e., $(x, y, z) = (0, 0, 0)$.*

Proof. Let x, y and z be non-negative integers with $x \neq 1$ such that (1.1) is true. We consider the following four cases:
Case 1. $x = 0$ and $y = 0$. From (1.1), we get $z = 0$. Then $(x, y, z) = (0, 0, 0)$.
Case 2. $x = 0$ and $y \geq 1$. From (1.1), we obtain $1 - p^y = z^2$ and so $z^2 < 0$, a contradiction.
Case 3. $x > 1$ and $y = 0$. From (1.1), we have $(p+a)^x - z^2 = 1$. If $z = 0$, then we get $(p+a)^x = 1$ and so $x = 0$, a contradiction. If $z = 1$, then $(p+a)^x = 2$. Thus $x = 1$ and $p + a = 2$, a contradiction. Therefore, $z > 1$, which is impossible, by Theorem 2.1.
Case 4. $x > 1$ and $y \geq 1$. Assume that x is odd. From (1.1), it follows that $a^x \equiv z^2 \pmod p$, which is impossible since $\left(\frac{a^x}{p}\right) = \left(\frac{a}{p}\right)^x = (-1)^x = -1$, by Theorem 2.8. Thus $x = 2k$ for some positive integer k. From (1.1), we have $[(p+a)^k - z][(p+a)^k + z] = p^y$. Since p is prime, there exists a non-negative integer u such that $(p+a)^k - z = p^u$ and $(p+a)^k + z = p^{y-u}$. Thus $2(p+a)^k = p^u[p^{y-2u}+1]$. Assume that $u > 0$. Since p is prime, we have $p \mid a$, which is impossible since $a \equiv 1 \pmod p$. Then $u = 0$. That is $2(p+a)^k = p^y + 1$ and so $2 \equiv 1 \pmod p$, which also is impossible. □

Theorem 3.4. *Let a be a positive integer with $a \equiv 2 \pmod 4$ and p be prime with $p \equiv 3 \pmod 4$. If $p+a$ is prime, $\gcd(p, a+1) = 1$ and $ord_p a = p-1$, then the Diophantine equation (1.2) has the unique non-negative integer solution; i.e. $(x, y, z) = (0, 0, 0)$.*

Proof. Let x, y and z be non-negative integers such that the equation (1.2) is true. We consider the following four cases:
Case 1. $x = 0$ and $y = 0$. From (1.2), we get $z = 0$. Then $(x, y, z) = (0, 0, 0)$.
Case 2. $x = 0$ and $y \geq 1$. From (1.2), we get $1-(p+a)^y = z^2$ and so $z^2 < 0$, a contradiction.
Case 3. $x \geq 1$ and $y = 0$. From (1.2), it follows that $p^x - z^2 = 1$. Assume that $x = 1$. Then $p = z^2 + 1$. Since $z^2 \equiv 0, 1 \pmod 4$, we get $z^2 + 1 \equiv 1, 2 \pmod 4$ and so $p \equiv 1, 2 \pmod 4$, a contradiction. Thus $x > 1$. It is easy to see that $z \notin \{0, 1\}$. Therefore $z > 1$, which is impossible, by Theorem 2.1.
Case 4. $x \geq 1$ and $y \geq 1$. From (1.2) and $z^2 \equiv 0, 1 \pmod 4$, we get $(-1)^x - 1 \equiv 0, 1 \pmod 4$. Thus $x = 2k$ for some positive integer k. From (1.2), we have $(p^k - z)(p^k + z) = (p+a)^y$. Since $p+a$ is prime, there exists a non-negative integer v such that $p^k - z = (p+a)^v$ and $p^k + z = (p+a)^{y-v}$. Thus $2p^k = (p+a)^v[(p+a)^{y-2v} + 1]$. Since p and $p+a$ are prime, we have $v = 0$. Then $2p^k = (p+a)^y + 1$ and so $a^y \equiv -1 \pmod p$. Assume that y is odd. Therefore, $2p^k = (p+a+1)[(p+a)^{y-1} - (p+a)^{y-2} + \cdots + 1]$ and so $p \mid (a+1)$, which is impossible since $\gcd(p, a+1) = 1$. Thus $y = 2h$ for some positive integer h. Therefore, $a^{2h} \equiv -1 \pmod p$. Since $ord_p a = p-1$ and Theorem 2.6, we have $a^{\frac{p-1}{2}} \equiv -1 \pmod p$. Then $a^{2h} \equiv a^{\frac{p-1}{2}} \pmod p$. By Theorem 2.5, we get $2h \equiv \frac{p-1}{2} \pmod{p-1}$. There exists an integer t such that $2h = (p-1)t + \frac{p-1}{2}$. Since $2h$ and $(p-1)t$ are even, we get $\frac{p-1}{2}$ is even and so $p \equiv 1 \pmod 4$, which is impossible since $p \equiv 3 \pmod 4$. □

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