International Journal of Mathematics and
Computer Science, **19**(2024), no. 3, 821–826

Generalized (τ_1, τ_2)-closed sets in bitopological spaces

Chokchai Viriyapong[1], Supannee Sompong[2], Chawalit Boonpok[1]

[1]Mathematics and Applied Mathematics Research Unit
Department of Mathematics
Faculty of Science
Mahasarakham University
Maha Sarakham, 44150, Thailand

[2]Department of Mathematics and Statistics
Faculty of Science and Technology
Sakon Nakhon Rajbhat University
Sakon Nakhon, 47000, Thailand

email: chokchai.v@msu.ac.th, s_sompong@snru.ac.th, chawalit.b@msu.ac.th

(Received December 14, 2023, Accepted February 1, 2024,
Published February 12, 2024)

Abstract

In this paper, we deal with the notion of generalized (τ_1, τ_2)-closed sets. First, we introduce the notion of generalized (τ_1, τ_2)-closed sets. Next, we study some properties of generalized (τ_1, τ_2)-closed sets. Finally, we investigate some properties of generalized (τ_1, τ_2)-open sets.

1 Introduction

Levine [6] introduced the notion of generalized closed sets in topological spaces and defined the notion of a $T_{\frac{1}{2}}$-space to be one in which the closed sets and the generalized closed sets coincide. Dunham and Levine [8] studied further properties of generalized closed sets. The notion of generalized closed

Key words and phrases: generalized (τ_1, τ_2)-closed set, generalized (τ_1, τ_2)-open set.
AMS (MOS) Subject Classifications: 54A05, 54E55.
The corresponding author is Chokchai Viriyapong.

sets has been modified and studied by using weaker forms of open sets such as semi-open sets, preopen sets, α-open sets and β-open sets. Dungthaisong et al. [7] investigated the notion of generalized closed sets in bigeneralized topological spaces and studied some characterizations of pairwise μ-$T_{\frac{1}{2}}$ spaces. Viriyapong and Boonpok [9] introduced and investigated the notion of generalized (Λ, p)-closed sets. Furthermore, some properties of generalized (Λ, α)-closed sets, generalized $\delta p(\Lambda, s)$-closed sets, generalized (Λ, s)-closed sets and generalized (Λ, sp)-closed sets were studied in [1], [2], [3] and [4], respectively. In this paper, we introduce the notion of generalized (τ_1, τ_2)-closed sets. Moreover, we investigate some properties of generalized (τ_1, τ_2)-closed sets and generalized (τ_1, τ_2)-open sets.

2 Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [5] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [5] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [5] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 2.1. [5] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ *and* $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, *then* $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ *is* $\tau_1\tau_2$*-closed.*

(4) A *is* $\tau_1\tau_2$*-closed if and only if* $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

3 Generalized (τ_1, τ_2)-closed sets in bitopological spaces

In this section, we introduce the notion of generalized (τ_1, τ_2)-closed sets. Moreover, we discuss some properties of generalized (τ_1, τ_2)-closed sets and generalized (τ_1, τ_2)-open sets.

Definition 3.1. *A subset A of a bitopological space (X, τ_1, τ_2) is said to be generalized (τ_1, τ_2)-closed (briefly, g-(τ_1, τ_2)-closed) if $\tau_1\tau_2$-Cl$(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_1\tau_2$-open.*

Theorem 3.2. *Let (X, τ_1, τ_2) be a bitopological space and $A, B \subseteq X$. If A and B are g-(τ_1, τ_2)-closed sets, then $A \cup B$ is g-(τ_1, τ_2)-closed.*

Proof. Let W be a $\tau_1\tau_2$-open set and $A \cup B \subseteq W$. Then $A \subseteq W$ and $B \subseteq W$. Since A and B are g-(τ_1, τ_2)-closed, we have $\tau_1\tau_2$-Cl$(A) \subseteq W$ and $\tau_1\tau_2$-Cl$(B) \subseteq W$. Thus $\tau_1\tau_2$-Cl$(A \cup B) = \tau_1\tau_2$-Cl$(A) \cup \tau_1\tau_2$-Cl$(B) \subseteq W$ and hence $A \cup B$ is g-(τ_1, τ_2)-closed. □

Theorem 3.3. *Let (X, τ_1, τ_2) be a bitopological space. If A is a g-(τ_1, τ_2)-closed set and F is a $\tau_1\tau_2$-closed set of X, then $A \cap F$ is g-(τ_1, τ_2)-closed.*

Proof. Let V be a $\tau_1\tau_2$-open set and $A \cap F \subseteq V$. Then $A \subseteq V \cup (X - F)$. Since A is g-(τ_1, τ_2)-closed and $V \cup (X - F)$ is $\tau_1\tau_2$-open, $\tau_1\tau_2$-Cl$(A) \subseteq V \cup (X - F)$. Thus $\tau_1\tau_2$-Cl$(A \cap F) \subseteq \tau_1\tau_2$-Cl$(A) \cap F \subseteq V$ and hence $A \cap F$ is g-(τ_1, τ_2)-closed. □

Theorem 3.4. *A subset A of a bitopological space (X, τ_1, τ_2) is g-(τ_1, τ_2)-closed if and only if $\tau_1\tau_2$-Cl$(A) - A$ contains no nonempty $\tau_1\tau_2$-closed set.*

Proof. Let F be a $\tau_1\tau_2$-closed subset of $\tau_1\tau_2$-Cl$(A) - A$. Then $A \subseteq X - F$. Since $X - F$ is $\tau_1\tau_2$-open and A is g-(τ_1, τ_2)-closed, $\tau_1\tau_2$-Cl$(A) \subseteq X - F$ and hence $F \subseteq X - \tau_1\tau_2$-Cl$(A)$. Thus $F \subseteq \tau_1\tau_2$-Cl$(A) \cap [X - \tau_1\tau_2$-Cl$(A)] = \emptyset$ and F is empty.

Conversely, suppose that $A \subseteq U$ and U is $\tau_1\tau_2$-open. If $\tau_1\tau_2$-Cl$(A) \nsubseteq U$, then $\tau_1\tau_2$-Cl$(A) \cap (X - U)$ is a nonempty $\tau_1\tau_2$-closed subset of

$$\tau_1\tau_2\text{-Cl}(A) - A.$$

□

Corollary 3.5. *Let A be a g-(τ_1, τ_2)-closed set of a bitopological space (X, τ_1, τ_2). Then A is $\tau_1\tau_2$-closed if and only if $\tau_1\tau_2$-Cl$(A) - A$ is $\tau_1\tau_2$-closed.*

Proof. If A is a $\tau_1\tau_2$-closed set, then $\tau_1\tau_2$-Cl$(A) - A = \emptyset$.

Conversely, suppose that $\tau_1\tau_2$-Cl$(A) - A$ is $\tau_1\tau_2$-closed. Since A is g-(τ_1, τ_2)-closed and $\tau_1\tau_2$-Cl$(A) - A$ is a $\tau_1\tau_2$-closed subset of itself, by Theorem 3.4, $\tau_1\tau_2$-Cl$(A) - A = \emptyset$ and hence $\tau_1\tau_2$-Cl$(A) = A$. □

Theorem 3.6. *A subset A of a bitopological space (X, τ_1, τ_2) is g-(τ_1, τ_2)-closed if and only if $\tau_1\tau_2$-Cl$(A) \cap F = \emptyset$, whenever $A \cap F = \emptyset$ and F is $\tau_1\tau_2$-closed.*

Proof. Suppose that A is a g-(τ_1, τ_2)-closed set. Let F be a $\tau_1\tau_2$-closed set and $A \cap F = \emptyset$. Then $A \subseteq X - F$. Since A is g-(τ_1, τ_2)-closed and $X - F$ is $\tau_1\tau_2$-open, $\tau_1\tau_2$-Cl$(A) \subseteq X - F$. Thus $\tau_1\tau_2$-Cl$(A) \cap F = \emptyset$.

Conversely, let U be $\tau_1\tau_2$-open and $A \subseteq U$. Then $A \cap (X - U) = \emptyset$ and $X - U$ is $\tau_1\tau_2$-closed. By the hypothesis, $\tau_1\tau_2$-Cl$(A) \cap (X - U) = \emptyset$ and hence $\tau_1\tau_2$-Cl$(A) \subseteq U$. Thus A is g-(τ_1, τ_2)-closed. □

Theorem 3.7. *A subset A of a bitopological space (X, τ_1, τ_2) is g-(τ_1, τ_2)-closed if and only if $\tau_1\tau_2$-Cl$(\{x\}) \cap A \neq \emptyset$ for each $x \in \tau_1\tau_2$-Cl(A).*

Proof. Let A be g-(τ_1, τ_2)-closed and suppose that there exists $x \in \tau_1\tau_2$-Cl(A) such that $\tau_1\tau_2$-Cl$(\{x\}) \cap A = \emptyset$. Thus $A \subseteq X - \tau_1\tau_2$-Cl$(\{x\})$ and hence $\tau_1\tau_2$-Cl$(A) \subseteq X - \tau_1\tau_2$-Cl$(\{x\})$. Therefore, $x \notin \tau_1\tau_2$-Cl(A), which is a contradiction.

Conversely, suppose that the condition of the theorem holds and let U be any $\tau_1\tau_2$-open set containing A. Let $x \in \tau_1\tau_2$-Cl(A). By the hypothesis, $\tau_1\tau_2$-Cl$(\{x\}) \cap A \neq \emptyset$ and so there exists $z \in \tau_1\tau_2$-Cl$(\{x\}) \cap A$. Hence $z \in A \subseteq U$. Thus $\{x\} \cap U \neq \emptyset$. Therefore, $x \in U$, which implies that $\tau_1\tau_2$-Cl$(A) \subseteq U$. This shows that A is g-(τ_1, τ_2)-closed. □

Definition 3.8. *Let (X, τ_1, τ_2) be a bitopological space and $A \subseteq X$. The (τ_1, τ_2)-frontier of A, (τ_1, τ_2)-$fr(A)$, is defined as follows:*

$$(\tau_1, \tau_2)\text{-}fr(A) = \tau_1\tau_2\text{-}Cl(A) \cap \tau_1\tau_2\text{-}Cl(X - A).$$

Theorem 3.9. *Let (X, τ_1, τ_2) be a bitopological space and let A be a g-(τ_1, τ_2)-closed set of X. If U is $\tau_1\tau_2$-open in X and $A \subseteq U$, then (τ_1, τ_2)-$fr(U) \subseteq \tau_1\tau_2$-Int$(X - A)$.*

Proof. Let U be a $\tau_1\tau_2$-open set and $A \subseteq U$. Then $\tau_1\tau_2$-Cl$(A) \subseteq U$. Suppose that $x \in (\tau_1, \tau_2)$-$fr(U)$. Since U is $\tau_1\tau_2$-open, (τ_1, τ_2)-$fr(U) = \tau_1\tau_2$-Cl$(U) - U$. Therefore, $x \notin U$ and $x \notin \tau_1\tau_2$-Cl(A). Thus $x \in \tau_1\tau_2$-Int$(X - A)$ and hence (τ_1, τ_2)-$fr(U) \subseteq \tau_1\tau_2$-Int$(X - A)$. □

Definition 3.10. *A subset A of a bitopological space (X, τ_1, τ_2) is said to be generalized (τ_1, τ_2)-open (briefly, g-(τ_1, τ_2)-open) if $X - A$ is generalized (τ_1, τ_2)-closed.*

Theorem 3.11. *Let (X, τ_1, τ_2) be a bitopological space and $A \subseteq X$. Then A is g-(τ_1, τ_2)-open if and only if $F \subseteq \tau_1\tau_2$-Int(A) whenever $F \subseteq A$ and F is $\tau_1\tau_2$-closed.*

Proof. Suppose that A is a g-(τ_1, τ_2)-open set. Let F be a $\tau_1\tau_2$-closed set and $F \subseteq A$. Then $X - A \subseteq X - F$. Since $X - A$ is g-(τ_1, τ_2)-closed and $X - F$ is $\tau_1\tau_2$-open, $\tau_1\tau_2$-Cl$(X - A) \subseteq X - F$. Thus $X - \tau_1\tau_2$-Int$(A) = \tau_1\tau_2$-Cl$(X - A) \subseteq X - F$ and hence $F \subseteq \tau_1\tau_2$-Int(A).

Conversely, let $X - A \subseteq U$ and U be $\tau_1\tau_2$-open. Then $X - U \subseteq A$. Since A is g-(τ_1, τ_2)-open and $X - U$ is $\tau_1\tau_2$-closed, $X - U \subseteq \tau_1\tau_2$-Int$(A)$. Therefore, $\tau_1\tau_2$-Cl$(X - A) = X - \tau_1\tau_2$-Int$(A) \subseteq U$. Thus $X - A$ is g-(τ_1, τ_2)-closed and hence A is g-(τ_1, τ_2)-open. □

Theorem 3.12. *Let (X, τ_1, τ_2) be a bitopological space and $A, B \subseteq X$. If A and B are g-(τ_1, τ_2)-open sets such that $\tau_1\tau_2$-Cl$(B) \cap A = \emptyset$ and*

$$\tau_1\tau_2\text{-}Cl(A) \cap B = \emptyset,$$

then $A \cup B$ is g-(τ_1, τ_2)-open.

Proof. Let F be a $\tau_1\tau_2$-closed subset of $A \cup B$. Then $\tau_1\tau_2$-Cl$(A) \cap F \subseteq A$ and hence, by Theorem 3.11, $\tau_1\tau_2$-Cl$(A) \cap F \subseteq \tau_1\tau_2$-Int$(A)$. Similarly, we have $\tau_1\tau_2$-Cl$(B) \cap F \subseteq \tau_1\tau_2$-Int$(B)$. Thus

$$\begin{aligned} F = (A \cup B) \cap F &\subseteq (\tau_1\tau_2\text{-Cl}(A)) \cup (\tau_1\tau_2\text{-Cl}(B)) \\ &\subseteq \tau_1\tau_2\text{-Int}(A) \cup \tau_1\tau_2\text{-Int}(B) = \tau_1\tau_2\text{-Int}(A \cup B). \end{aligned}$$

By Theorem 3.11, $A \cup B$ is g-(τ_1, τ_2)-open. □

Theorem 3.13. *Let (X, τ_1, τ_2) be a bitopological space. For each $x \in X$, $\{x\}$ is either $\tau_1\tau_2$-closed or g-(τ_1, τ_2)-open.*

Proof. Suppose that $\{x\}$ is not $\tau_1\tau_2$-closed. Then $X - \{x\}$ is not $\tau_1\tau_2$-open and the only $\tau_1\tau_2$-open set containing $X - \{x\}$ is X itself. Thus $\tau_1\tau_2$-Cl$(X - \{x\}) \subseteq X$ and hence $X - \{x\}$ is g-(τ_1, τ_2)-closed. This shows that $\{x\}$ is g-(τ_1, τ_2)-open. □

Acknowledgment. This research project was financially supported by Mahasarakham University.

References

[1] C. Boonpok, M. Thongmoon, Some properties of generalized (Λ, α)-closed sets, Int. J. Anal. Appl., **21,** (2023), 88.

[2] C. Boonpok, N. Srisarakham, Properties of generalized $\delta p(\Lambda, s)$-closed sets, Eur. J. Pure Appl. Math., **16,** no. 4, (2023), 2581–2596.

[3] C. Boonpok, C. Viriyapong, On some forms of closed sets and related topics, Eur. J. Pure Appl. Math., **16,** no. 1, (2023), 336–362.

[4] C. Boonpok, C. Viriyapong, On generalized (Λ, sp)-closed sets, Eur. J. Pure Appl. Math., **15,** no. 4, (2022), 2127–2140.

[5] C. Boonpok, C. Viriyapong, M. Thongmoon, On upper and lower (τ_1, τ_2)-precontinuous multifunctions, J. Math. Computer Sci., **18z,** (2018), 282–293.

[6] N. Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo, **19,** (1970), 89–96.

[7] W. Dungthaisong, C. Boonpok, C. Viriyapong, Generalized closed sets in bigeneralized topological spaces, Int. J. Math. Anal., **5,** no. 24, (2011), 1175–1184.

[8] W. Dunham, N. Levine, Further results on generalized closed sets, Kyungpook Math. J., **20,** (1980), 169–175.

[9] C. Viriyapong, C. Boonpok, On generalized (Λ, p)-closed sets, Int. J. Math. Comput. Sci., **18,** no. 1, (2023), 79–83.