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On some characterizations of (τ_1, τ_2)-$T_{\frac{1}{2}}$-spaces

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Abstract

In this paper, we introduce the notion of (τ_1, τ_2)-$T_{\frac{1}{2}}$-spaces. We also investigate some characterizations of (τ_1, τ_2)-$T_{\frac{1}{2}}$-spaces.

1 Introduction

In 1970, Levine [12] introduced the notion of generalized closed sets in topological spaces and defined a class of topological spaces called $T_{\frac{1}{2}}$-spaces; a topological space (X, τ) is $T_{\frac{1}{2}}$ if every generalized closed set is closed. Dunham [10] showed that a topological space (X, τ) is $T_{\frac{1}{2}}$ if and only if each singleton of X is open or closed. Arenas et al. [1] proved that a topological

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space (X, τ) is $T_{\frac{1}{2}}$ if and only if every subset of X is λ-closed. Dontchev and Ganster [8] introduced the notion of $T_{\frac{3}{4}}$-spaces which are situated between T_1 and $T_{\frac{1}{2}}$ and showed that the digital line or the Khalimsky line [11] $(\mathbb{Z}, \kappa)$ lies between T_1 and $T_{\frac{3}{4}}$. In 2012, Dungthaisong et al. [9] introduced and studied the notion of pairwise μ-$T_{\frac{1}{2}}$ spaces. Torton et al. [13] introduced and investigated the notions of $\mu_{(m,n)}$-T_1 spaces, $\mu_{(m,n)}$-T_3 spaces and $\mu_{(m,n)}$-T_4 spaces. Buadong et al. [6] introduced and studied the notions of T_1-GTMS spaces and T_2-GTMS spaces. Recently, Viriyapong and Boonpok [15] investigated several characterizations of (Λ, p)-$T_{\frac{1}{2}}$-spaces. Moreover, some characterizations $\delta p(\Lambda, s)$-$T_{\frac{1}{2}}$-spaces, Λ_α-$T_{\frac{1}{2}}$-spaces and (Λ, s)-$T_{\frac{1}{2}}$-spaces were established in [2], [3] and [4], respectively. In this paper, we introduce the notion of (τ_1, τ_2)-$T_{\frac{1}{2}}$-spaces and investigate several characterizations of (τ_1, τ_2)-$T_{\frac{1}{2}}$-spaces.

2 Preliminaries

Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called *$\tau_1\tau_2$-closed* [5] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called *$\tau_1\tau_2$-open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the *$\tau_1\tau_2$-closure* [5] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the *$\tau_1\tau_2$-interior* [5] of A and is denoted by $\tau_1\tau_2$-Int(A). The set $\cap\{G \mid A \subseteq G$ and G is $\tau_1\tau_2$-open$\}$ is called the *$\tau_1\tau_2$-kernel* [5] of A and is denoted by $\tau_1\tau_2$-$ker(A)$. A subset A of a bitopological space (X, τ_1, τ_2) is called a *$\Lambda_{(\tau_1,\tau_2)}$-set* [7] if $A = \tau_1\tau_2$-$ker(A)$.

Lemma 2.1. [5] *For subsets A, B of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) $A \subseteq \tau_1\tau_2$-$ker(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2$-$ker(A) \subseteq \tau_1\tau_2$-$ker(B)$.

(3) If A is $\tau_1\tau_2$-open, then $\tau_1\tau_2$-$ker(A) = A$.

(4) $x \in \tau_1\tau_2$-$ker(A)$ if and only if $A \cap H \neq \emptyset$ for every $\tau_1\tau_2$-closed set H containing x.

3 Some characterizations of (τ_1, τ_2)-$T_{\frac{1}{2}}$-spaces

In this section, we introduce the notion of (τ_1, τ_2)-$T_{\frac{1}{2}}$-spaces. Moreover, we discuss some characterizations of (τ_1, τ_2)-$T_{\frac{1}{2}}$-spaces.

Definition 3.1. [14] *A subset A of a bitopological space (X, τ_1, τ_2) is said to be generalized (τ_1, τ_2)-closed (briefly, g-(τ_1, τ_2)-closed) if $\tau_1\tau_2$-$Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_1\tau_2$-open.*

Definition 3.2. *A bitopological space (X, τ_1, τ_2) is called (τ_1, τ_2)-$T_{\frac{1}{2}}$ if every g-(τ_1, τ_2)-closed set of X is $\tau_1\tau_2$-closed.*

Definition 3.3. [7] *A subset A of a bitopological space (X, τ_1, τ_2) is called a $\Lambda^{\star}_{(\tau_1,\tau_2)}$-set if $\tau_1\tau_2$-$ker(A) \subseteq F$ whenever $A \subseteq F$ and F is $\tau_1\tau_2$-closed.*

Lemma 3.4. *For a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) for each $x \in X$, the singleton $\{x\}$ is $\tau_1\tau_2$-closed or $X - \{x\}$ is g-(τ_1, τ_2)-closed;

(2) for each $x \in X$, the singleton $\{x\}$ is $\tau_1\tau_2$-open or $X - \{x\}$ is a $\Lambda^{\star}_{(\tau_1,\tau_2)}$-set.

Proof. (1) Let $x \in X$ and the singleton $\{x\}$ be not $\tau_1\tau_2$-closed. Then we have $X - \{x\}$ is not $\tau_1\tau_2$-open and X is the only $\tau_1\tau_2$-open set which contains $X - \{x\}$ and hence $X - \{x\}$ is g-(τ_1, τ_2)-closed.

(2) Let $x \in X$ and the singleton $\{x\}$ be not $\tau_1\tau_2$-open. Then we have $X - \{x\}$ is not $\tau_1\tau_2$-closed and the only $\tau_1\tau_2$-closed set which contains $X - \{x\}$ is X and hence $X - \{x\}$ is a $\Lambda^{\star}_{(\tau_1,\tau_2)}$-set. □

Theorem 3.5. *For a topological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) (X, τ_1, τ_2) is (τ_1, τ_2)-$T_{\frac{1}{2}}$;

(2) for each $x \in X$, the singleton $\{x\}$ is $\tau_1\tau_2$-open or $\tau_1\tau_2$-closed;

(3) every $\Lambda^{\star}_{(\tau_1,\tau_2)}$-set is a $\Lambda_{(\tau_1,\tau_2)}$-set.

Proof. (1) $\Rightarrow$ (2): By Lemma 3.4, for each $x \in X$, the singleton $\{x\}$ is $\tau_1\tau_2$-closed or $X - \{x\}$ is g-(τ_1, τ_2)-closed. Since (X, τ_1, τ_2) is a (τ_1, τ_2)-$T_{\frac{1}{2}}$-space, $X - \{x\}$ is $\tau_1\tau_2$-closed and hence $\{x\}$ is $\tau_1\tau_2$-open in the latter case. Thus the singleton $\{x\}$ is $\tau_1\tau_2$-open or $\tau_1\tau_2$-closed.

(2) $\Rightarrow$ (3): Suppose that there exists a $\Lambda^{\star}_{(\tau_1,\tau_2)}$-set A which is not a $\Lambda_{(\tau_1,\tau_2)}$-set. There exists $x \in \tau_1\tau_2$-$ker(A)$ such that $x \notin A$. In case the singleton $\{x\}$ is $\tau_1\tau_2$-open, $A \subseteq X - \{x\}$ and $X - \{x\}$ is $\tau_1\tau_2$-closed. Since A is a $\Lambda^{\star}_{(\tau_1,\tau_2)}$-set, $\tau_1\tau_2$-$ker(A) \subseteq X - \{x\}$. This is a contradiction. In case the singleton $\{x\}$ is $\tau_1\tau_2$-closed, $A \subseteq X - \{x\}$ and $X - \{x\}$ is $\tau_1\tau_2$-open. By Lemma 2.1, $\tau_1\tau_2$-$ker(A) \subseteq \tau_1\tau_2$-$ker(X - \{x\}) = X - \{x\}$. This is a contradiction. Thus every $\Lambda^{\star}_{(\tau_1,\tau_2)}$-set is a $\Lambda_{(\tau_1,\tau_2)}$-set.

(3) $\Rightarrow$ (1): Suppose that (X, τ_1, τ_2) is not a (τ_1, τ_2)-$T_{\frac{1}{2}}$-space. Then there exists a g-(τ_1, τ_2)-closed set A which is not $\tau_1\tau_2$-closed. Since A is not $\tau_1\tau_2$-closed, there exists $x \in \tau_1\tau_2$-Cl(A) such that $x \notin A$. By Lemma 3.4, the singleton $\{x\}$ is $\tau_1\tau_2$-open or $X - \{x\}$ is a $\Lambda^{\star}_{(\tau_1,\tau_2)}$-set. (a) In case $\{x\}$ is $\tau_1\tau_2$-open, since $x \in \tau_1\tau_2$-Cl(A), $\{x\} \cap A \neq \emptyset$ and $x \in A$. This is a contradiction. (b) In case $X - \{x\}$ is a $\Lambda^{\star}_{(\tau_1,\tau_2)}$-set, if $\{x\}$ is not $\tau_1\tau_2$-closed, $X - \{x\}$ is not $\tau_1\tau_2$-open and $\tau_1\tau_2$-$ker(X - \{x\}) = X$. Hence, $X - \{x\}$ is not a $\Lambda^{\star}_{(\tau_1,\tau_2)}$-set. This contradicts (3). If $\{x\}$ is $\tau_1\tau_2$-closed, then $X - \{x\}$ is $\tau_1\tau_2$-open. Since $A \subseteq X - \{x\}$ and A is g-(τ_1, τ_2)-closed, we have $\tau_1\tau_2$-Cl$(A) \subseteq X - \{x\}$. This contradicts that $x \in \tau_1\tau_2$-Cl(A). Therefore, (X, τ_1, τ_2) is (τ_1, τ_2)-$T_{\frac{1}{2}}$. □

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