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On regular generalized (τ_1, τ_2)-closed sets

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Abstract

In this paper, we deal with the concept of regular generalized (τ_1, τ_2)-closed sets. First, we introduce the notion of regular generalized (τ_1, τ_2)-closed sets. Next, we study some properties of regular generalized (τ_1, τ_2)-closed sets and regular generalized (τ_1, τ_2)-open sets. Finally, we consider some characterizations of (τ_1, τ_2)-$T^{\star}_{\frac{1}{2}}$-spaces.

1 Introduction

Levine [10] introduced generalized closed sets and generalized open sets in topological spaces. Dunham and Levine [9] investigated further properties of generalized closed sets. Noiri and Roy [12] introduced and studied the

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concept of generalized μ-closed sets in a topological space by using the concept of generalized open sets introduced by Császár [7]. The class of all generalized μ-closed sets is strictly larger than the class of all μ-closed sets. Furthermore, generalized closed sets is a special type of generalized μ-closed sets in a topological space. Dungthaisong et al. [8] introduced and studied the notions of $\mu_{(m,n)}$-closed sets and $\mu_{(m,n)}$-open sets in bigeneralized topological spaces. Some properties of generalized (Λ, α)-closed sets, generalized $\delta p(\Lambda, s)$-closed sets, generalized (Λ, s)-closed sets, generalized (Λ, sp)-closed sets and generalized (Λ, p)-closed sets were studied in [1], [2], [3], [4] and [15], respectively. As a modification of generalized closed sets, Palaniappan and Rao [13] introduced and studied the notion of regular generalized closed sets. As a further modification of regular generalized closed sets, Noiri and Popa [11] introduced and investigated the concept of regular generalized α-closed sets. Roy [14] defined a new kind of sets called regular μ-generalized closed sets in a topological space. In this paper, we introduce the concept of regular generalized (τ_1, τ_2)-closed sets. Moreover, we investigate some properties of regular generalized (τ_1, τ_2)-closed sets and regular generalized (τ_1, τ_2)-open sets.

2 Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [6] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [6] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [6] of A and is denoted by $\tau_1\tau_2$-Int(A). A subset A of a bitopological space (X, τ_1, τ_2) is called $(\tau_1, \tau_2)r$-*open* (resp. $(\tau_1, \tau_2)r$-*closed*) [16] if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A = \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$).

Lemma 2.1. [6] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2$-Cl(A) and $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Cl$(A)) = \tau_1\tau_2$-Cl(A).

(2) If $A \subseteq B$, then $\tau_1\tau_2$-Cl$(A) \subseteq \tau_1\tau_2$-Cl$(B)$.

(3) $\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl(A).

(5) $\tau_1\tau_2$-Cl$(X - A) = X - \tau_1\tau_2$-Int$(A)$.

3 On regular generalized (τ_1, τ_2)-closed sets

We begin this section by introducing the concept of regular generalized (τ_1, τ_2)-closed sets.

Definition 3.1. *A subset A of a bitopological space (X, τ_1, τ_2) is said to be regular generalized (τ_1, τ_2)-closed (briefly, rg-(τ_1, τ_2)-closed) if $\tau_1\tau_2$-Cl$(A) \subseteq U$ whenever $A \subseteq U$ and U is $(\tau_1, \tau_2)r$-open.*

Theorem 3.2. *Let (X, τ_1, τ_2) be a bitopological space and $A, B \subseteq X$. If A and B are rg-(τ_1, τ_2)-closed sets, then $A \cup B$ is rg-(τ_1, τ_2)-closed.*

Proof. Let W be a $(\tau_1, \tau_2)r$-open set and $A \cup B \subseteq W$. Then, $A \subseteq W$ and $B \subseteq W$. Since A and B are rg-(τ_1, τ_2)-closed, we have $\tau_1\tau_2$-Cl$(A) \subseteq W$ and $\tau_1\tau_2$-Cl$(B) \subseteq W$. Thus, $\tau_1\tau_2$-Cl$(A \cup B) = \tau_1\tau_2$-Cl$(A) \cup \tau_1\tau_2$-Cl$(B) \subseteq W$ and hence $A \cup B$ is rg-(τ_1, τ_2)-closed. □

Theorem 3.3. *Let (X, τ_1, τ_2) be a bitopological space and $A \subseteq X$. If A is rg-(τ_1, τ_2)-closed, then $\tau_1\tau_2$-Cl$(A) - A$ contains no nonempty $(\tau_1, \tau_2)r$-closed set.*

Proof. Suppose that A is rg-(τ_1, τ_2)-closed. Let F be a $(\tau_1, \tau_2)r$-closed subset of $\tau_1\tau_2$-Cl$(A) - A$. Then $F \subseteq \tau_1\tau_2$-Cl$(A) \cap (X - A)$ and hence $A \subseteq X - F$. Since $X - F$ is $\tau_1\tau_2$-open and A is rg-(τ_1, τ_2)-closed, $\tau_1\tau_2$-Cl$(A) \subseteq X - F$. Therefore, $F \subseteq X - \tau_1\tau_2$-Cl$(A)$. Since $F \subseteq \tau_1\tau_2$-Cl(A), $F \subseteq \tau_1\tau_2$-Cl$(A) \cap [X - \tau_1\tau_2$-Cl$(A)] = \emptyset$. This shows that $F = \emptyset$. □

Corollary 3.4. *Let (X, τ_1, τ_2) be a bitopological space and A be a rg-(τ_1, τ_2)-closed set. Then A is $(\tau_1, \tau_2)r$-closed if and only if $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)) - A$ is $(\tau_1, \tau_2)r$-closed.*

Proof. Let A be a $(\tau_1, \tau_2)r$-closed set. Then, $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)) - A = \emptyset$. Thus, $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)) - A$ is $(\tau_1, \tau_2)r$-closed.

Conversely, suppose that $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)) - A$ is $(\tau_1, \tau_2)r$-closed. Since A is rg-(τ_1, τ_2)-closed and $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)) - A$ contains the $(\tau_1, \tau_2)r$-closed set $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)) - A$. By Theorem 3.3,

$$\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)) - A = \emptyset.$$

Thus, $\tau_1\tau_2$-Cl($\tau_1\tau_2$-Int$(A)) = A$ and hence A is $(\tau_1, \tau_2)r$-closed. □

Theorem 3.5. *Let (X, τ_1, τ_2) be a bitopological space and $A \subseteq X$. If A is rg-(τ_1, τ_2)-closed and $A \subseteq B \subseteq \tau_1\tau_2$-Cl$(A)$, then $\tau_1\tau_2$-Cl$(B) - B$ contains no nonempty $(\tau_1, \tau_2)r$-closed set.*

Proof. $A \subseteq B$ implies $X - B \subseteq X - A$ and $B \subseteq \tau_1\tau_2$-Cl(A) implies

$$\tau_1\tau_2\text{-Cl}(B) \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Cl}(A)) = \tau_1\tau_2\text{-Cl}(A).$$

Thus, $\tau_1\tau_2$-Cl$(B) \subseteq \tau_1\tau_2$-Cl$(A)$ and hence $\tau_1\tau_2$-Cl$(B) - B \subseteq \tau_1\tau_2$-Cl$(A) - A$. Since A is rg-(τ_1, τ_2)-closed, $\tau_1\tau_2$-Cl$(A) - A$ has no nonempty $(\tau_1, \tau_2)r$-closed subsets, neither does $\tau_1\tau_2$-Cl$(B) - B$. □

Definition 3.6. *A subset A of a bitopological space (X, τ_1, τ_2) is said to be regular generalized (τ_1, τ_2)-open (briefly, rg-(τ_1, τ_2)-open) if $X - A$ is regular generalized (τ_1, τ_2)-closed.*

Theorem 3.7. *A subset A of a bitopological space (X, τ_1, τ_2) is rg-(τ_1, τ_2)-open if and only if $F \subseteq \tau_1\tau_2$-Int(A) whenever $F \subseteq A$ and F is $(\tau_1, \tau_2)r$-closed.*

Proof. Suppose that A is a rg-(τ_1, τ_2)-open set. Let F be a $(\tau_1, \tau_2)r$-closed set and $F \subseteq A$. Then $X - A \subseteq X - F$. Since $X - A$ is rg-(τ_1, τ_2)-closed and $X - F$ is $(\tau_1, \tau_2)r$-open, $\tau_1\tau_2$-Cl$(X - A) \subseteq X - F$. Thus, $X - \tau_1\tau_2$-Int$(A) = \tau_1\tau_2$-Cl$(X - A) \subseteq X - F$ and hence $F \subseteq \tau_1\tau_2$-Int(A).

Conversely, let $X - A \subseteq U$ and U be $(\tau_1, \tau_2)r$-open. Then $X - U \subseteq A$. Since A is rg-(τ_1, τ_2)-open and $X - U$ is $(\tau_1, \tau_2)r$-closed, $X - U \subseteq \tau_1\tau_2$-Int$(A)$. This implies that $\tau_1\tau_2$-Cl$(X - A) = X - \tau_1\tau_2$-Int$(A) \subseteq U$. Thus $X - A$ is rg-(τ_1, τ_2)-closed and hence A is rg-(τ_1, τ_2)-open. □

Theorem 3.8. *Let (X, τ_1, τ_2) be a bitopological space and $A \subseteq X$. If A is rg-(τ_1, τ_2)-closed in X, then $W = X$ whenever W is $(\tau_1, \tau_2)r$-open and $\tau_1\tau_2$-Int$(A) \cup (X - A) \subseteq W$.*

Proof. Suppose that A is rg-(τ_1, τ_2)-closed in X. Let W be a $(\tau_1, \tau_2)r$-open set and $\tau_1\tau_2$-Int$(A) \cup (X - A) \subseteq W$. Then $X - W \subseteq [X - \tau_1\tau_2\text{-Int}(A)] \cap A$ and hence $X - W \subseteq [X - \tau_1\tau_2\text{-Int}(A)] - (X - A) = \tau_1\tau_2\text{-Cl}(X - A) - (X - A)$. Since $X - W$ is $(\tau_1, \tau_2)r$-closed and $X - A$ is rg-(τ_1, τ_2)-closed, by Theorem 3.3, $X - W = \emptyset$. Consequently, $X = W$. □

Theorem 3.9. *Let (X, τ_1, τ_2) be a bitopological space and let $A \subseteq X$. If A is rg-(τ_1, τ_2)-closed in X, then $\tau_1\tau_2$-Cl$(A) - A$ is rg-$(\tau_1, \tau_2)r$-open.*

Proof. Suppose that A is rg-(τ_1, τ_2)-closed. Let F be a $(\tau_1, \tau_2)r$-closed set and let $F \subseteq \tau_1\tau_2$-Cl$(A) - A$. Then, by Theorem 3.3, $F = \emptyset$ and hence

$$F \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A) - A).$$

By Theorem 3.7, $\tau_1\tau_2$-Cl$(A) - A$ is rg-$(\tau_1, \tau_2)r$-open. □

Definition 3.10. *A bitopological space* (X, τ_1, τ_2) *is called* (τ_1, τ_2)-$T^{\star}_{\frac{1}{2}}$ *if every* rg-(τ_1, τ_2)*-closed set of* X *is* $\tau_1\tau_2$*-closed.*

Theorem 3.11. *A bitopological space* (X, τ_1, τ_2) *is* (τ_1, τ_2)-$T^{\star}_{\frac{1}{2}}$ *if and only if every singleton of* X *is* $(\tau_1, \tau_2)r$*-closed or* $\tau_1\tau_2$*-open.*

Proof. Suppose that (X, τ_1, τ_2) is (τ_1, τ_2)-$T^{\star}_{\frac{1}{2}}$. Let $x \in X$. If $\{x\}$ is not $(\tau_1, \tau_2)r$-closed, then $X - \{x\}$ is not $(\tau_1, \tau_2)r$-open and hence X is the only $(\tau_1, \tau_2)r$-open set containing $X - \{x\}$. Thus $X - \{x\}$ is rg-(τ_1, τ_2)-closed. By the hypothesis, $X - \{x\}$ is $\tau_1\tau_2$-closed and so $\{x\}$ is $\tau_1\tau_2$-open.

Conversely, suppose that every singleton of X is $(\tau_1, \tau_2)r$-closed or $\tau_1\tau_2$-open. Let A be a rg-(τ_1, τ_2)-closed set of X and $x \in \tau_1\tau_2$-Cl(A). If $\{x\}$ is $\tau_1\tau_2$-open, then $\{x\} \cap A \neq \emptyset$. Therefore, $x \in A$. If $\{x\}$ is $r(\tau_1, \tau_2)$-closed, it follows from Theorem 3.3 that $x \notin \tau_1\tau_2$-Cl$(A) - A$ and so $x \in A$. Thus in the both cases, $x \in A$ and hence A is $\tau_1\tau_2$-closed. This shows that (X, τ_1, τ_2) is (τ_1, τ_2)-$T^{\star}_{\frac{1}{2}}$. □

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