International Journal of Mathematics and
Computer Science, **19**(2024), no. 3, 553–560

On the square of Fibonacci and Lucas numbers of the form $(2^{2s}-1)^x+(2^{s+1})^y$

Hunar Sherzad Taher, Saroj Kumar Dash

Mathematics Division
School of Advanced Science
Vellore Institute of Technology, Chennai Campus
Chennai-600127, India

email: hunarsherzad.taher2022@vitstudent.ac.in, sarojkumar.dash@vit.ac.in

(Received October 23, 2023, Accepted November 24, 2023,
Published February 12, 2024)

Abstract

Let F_k be a Fibonacci number and let L_k be a Lucas number. By applying Catalan's conjecture and the modular arithmetic method, we solve the exponential Diophantine equations of the form $(2^{2s}-1)^x+(2^{s+1})^y=F_k^2$ and $(2^{2s}-1)^x+(2^{s+1})^y=L_k^2$ where x,y,k are non-negative integers and s is a positive integer.

1 Introduction

In 2007, Acu [3] proved that the Diophantine equation $2^x+5^y=z^2$ has only two solutions $(3,0,3)$ and $(2,1,3)$ where x,y and z are non-negative integers. Later, many papers have been published on this type of Diophantine equation $p^x+q^y=z^2$. Suvarnamani [1] found the solutions of the Diophantine equation $2^x+p^y=z^2$, for non-negative integers x,y,z and prime number p. Sroysang [4] and Rabago [5] have done work on the Diophantine equations $3^x+5^y=z^2$ and $2^x+17^y=z^2$ respectively.
Recently, Elshahed, Kamarulhaili [2], Mina and Bacani [13], Borah and Dutta [6], Pakapongpun and Chattae [7], Tadee [8] and Tadee and Siraworakun [14]

Key words and phrases: Exponential Diophantine equation, Fibonacci number, Lucas number.
AMS (MOS) Subject Classifications: 11D61, 11D45, 11B39.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

have worked on $(4^n)^x - p^y = z^2$, $p^x + (p+4k)^y = z^2$, $7^x + 32^y = z^2$, $p^x + 7^y = z^2$, $n^x + 10^y = z^2$ and $p^x + (p+2q)^y = z^2$, respectively.
On the other hand, several mathematicians have extensively investigated Diophantine equations related to linear recurrence sequences. In 2016, Bravo and Luca [9] worked on solutions of the Diophantine equation $F_n + F_m = 2^a$. Marques and Togbé [10] found all solutions of the Fibonacci and Lucas numbers of the form $2^a + 3^b + 5^c$, where a, b, c are non-negative integers with $c \geq \max\{a, b\} \geq 0$. Moreover, Qu and Zeng [11] found all Pell and Pell-Lucas Numbers written in the form $-2^a - 3^b + 5^c$, where a, b, c are non-negative integers with $c \geq \max\{a, b\} \geq 0$. Using linear forms in logarithms of algebraic numbers and the Baker-Davenport reduction method, Tiebekabe and Diouf [12] found the solutions of the Diophantine equation $L_n + L_m = 3^a$.
In this paper, we find all solutions of the Diophantine equation

$$(2^{2s} - 1)^x + (2^{s+1})^y = z^2,$$

where x, y are non-negative integers, s is a positive integer and z is a Fibonacci or a Lucas number.

2 Preliminaries

Let $\{F_k\}_{k \geq 0}$ be the Fibonacci sequence defined by the recurrence relations

$$F_0 = 0, F_1 = 1, F_k = F_{k-1} + F_{k-2} \text{ for } k \geq 2.$$

Let $\{L_k\}_{k \geq 0}$ be the Lucas sequence defined by the recurrence relations

$$L_0 = 2, L_1 = 1, L_k = L_{k-1} + L_{k-2} \text{ for } k \geq 2.$$

The characteristic polynomial of both sequences is

$$f(x) = x^2 - x - 1,$$

which has the two roots $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$.
The Benit's formulas for the Fibonacci and Lucas sequences are defined by

$$F_k = \frac{\alpha^k - \beta^k}{\alpha - \beta}, \quad L_k = \alpha^k - \beta^k,$$

The Online Encyclopedia of Integer (OEIS) of Fibonacci and Lucas sequences are A000045 and A000032, respectively.

Theorem 2.1. *([15], Catalan's conjecture/Mihăilescu's theorem) The Diophantine equation* $a^x - b^y = 1$ *has exactly one solution* $(3, 2, 2, 3)$*, where* x, y, a *and* b *are integers with* $min\{a, b, x, y\} > 1$.

Lemma 2.2. *If* m *is an odd number, then* $m \equiv 1$ *or* $3 \pmod 4$.

Proof. Let m be an odd number. Then there exists $s \in \mathbb{Z}$ such that $m = 4s+1$ or $m = 4s + 3$.
Case 1. If $m = 4s + 1$, then we have $4|m - 1$ which implies that $m \equiv 1 \pmod 4$.
Case 2. If $m = 4s + 3$, then we have $4|m - 3$ which implies that $m \equiv 3 \pmod 4$. □

Lemma 2.3. *If* m *is an integer, then* $m^2 \equiv 0$ *or* $1 \pmod 4$.

Proof. Let m be an integer. We consider the following two exclusive cases:
Case 1. m is an even number. We can write $m = 2s$, where $s \in \mathbb{Z}$. Then $m^2 = 4s^2$ which implies that $4|m^2$ or $m^2 \equiv 0 \pmod 4$.
Case 2. m is an odd number. We can write $m = 2s + 1$, where $s \in \mathbb{Z}$. Then $m^2 = (2s + 1)^2 = 4s^2 + 4s + 1$. So $m^2 - 1 = 4(s^2 + s)$ which implies that $4|m^2 - 1$ or $m^2 \equiv 1 \pmod 4$. □

Lemma 2.4. *If* s *and* x *are positive integers, then* $(2^{2s} - 1)^x \equiv 1$ *or* $3 \pmod 4$.

Proof. Let s, x be two positive integers. We consider the following two exclusive cases:
Case 1. x is an even number. We can write $x = 2r$, where $r \in \mathbb{Z}^+$. We have $(2^{2s} - 1) \equiv -1 \pmod 4$, for each positive integer s. Then $(2^{2s} - 1)^{2r} \equiv 1 \pmod 4$, for each $r \in \mathbb{Z}^+$.
Case 2. x is an odd number. We can write $x = 2r+1$, where $r \in \mathbb{Z}$. We have $(2^{2s} - 1) \equiv -1 \pmod 4$, for each positive integer s. Then $(2^{2s} - 1)^{2r+1} \equiv -1 \pmod 4$, for each $r \in \mathbb{Z}^+$ which implies that $(2^{2s} - 1)^{2r+1} \equiv 3 \pmod 4$. □

Lemma 2.5. *If* s *and* y *are positive integers, then* $(2^{s+1})^y \equiv 0 \pmod 4$.

Proof. Let s, y be two positive integers. We have $(2^{s+1}) \equiv 0 \pmod 4$. Then $(2^{s+1})^y \equiv 0 \pmod 4$, for each positive integer s and y. □

3 Main Results

Theorem 3.1. *If F_k is a Fibonacci number, then the solutions of the Diophantine equation*

$$(2^{2s}-1)^x + (2^{s+1})^y = F_k^2, \tag{3.1}$$

are $(s, x, y, F_k) \in \{(1,1,0,2), (2,0,1,3), (1,2,2,5), (3,1,0,8)\}$, where s is a positive integer and x, y, k are non-negative integers.

Proof. We consider the following exclusive cases for all non-negative integers x, y and k:
Case 1. If $k = 0$, then we have $F_0 = 0$. This implies that $(2^{2s}-1)^x + (2^{s+1})^y = 0$ which is impossible for any value of x and y.
Case 2. If $x = 0$, then $1 + (2^{s+1})^y = F_k^2$ which, by Theorem 2.1, has only one solution $(s, x, y, F_k) = (2, 0, 1, 3)$.
Case 3. If $y = 0$, then $(2^{2s}-1)^x + 1 = F_k^2$ is solvable only when $(s, x, y, F_k) \in \{(1,1,0,2), (3,1,0,8)\}$.
Case 4. If $x = y = 0$, then we have $1 + 1 = F_k^2$ which is impossible.
Case 5. If $x = y = k = 0$, then we get $1 + 1 = 0$ which is impossible.
For the remaining possibilities, we can now assume that x, y, k are positive integers.
Case 6. If F_k is an even number, then $k = 3j$, for each $j \in \mathbb{Z}^+$. By Lemma 2.3, we have $F_k^2 \equiv 0 \pmod 4$, where $k = 3j$, for each $j \in \mathbb{Z}^+$. According to Lemmas 2.4 and 2.5, we have $(2^{2s}-1)^x + (2^{s+1})^y \equiv 1 \text{ or } 3 \pmod 4$ for each positive integer y and $x = 2r$ or $2r+1$, for each $r \in \mathbb{Z}^+$ which is impossible by our assumption.
Case 7. Let F_k be an odd number. Then $k = 3j-2$ or $3j-1$, for each $j \in \mathbb{Z}^+$. By Lemma 2.2, we have $F_k \equiv 1 \text{ or } 3 \pmod 4$, where $k = 3j-2$ or $3j-1$, for each $j \in \mathbb{Z}^+$. By Lemma 2.3, we get $F_k^2 \equiv 1 \pmod 4$, where $k = 3j-2$ or $3j-1$ for each $j \in \mathbb{Z}^+$.
Also, by Lemmas 2.4 and 2.5, we have

$$(2^{2s}-1)^x + (2^{s+1})^y \equiv 1 \pmod 4,$$

for each positive integer y and $x = 2r$ and for each $r \in \mathbb{Z}^+$. Thus

$$(2^{2s}-1)^{2r} + (2^{s+1})^y = F_k^2,$$

$$(2^{s+1})^y = (F_k + (2^{2s}-1)^r)(F_k - (2^{2s}-1)^r),$$

there exist non-negative integers α and β, where $\alpha + \beta = (s+1)y$ and $\alpha > \beta$, such that $2^\alpha = F_k + (2^{2s}-1)^r$ and $2^\beta = F_k - (2^{2s}-1)^r$. As a result,

$$2(2^{2s}-1)^r = 2^\beta(2^{\alpha-\beta}-1),$$

which implies that $2^\beta = 2$. Thus $\beta = 1$. Therefore,

$$(2^{2s} - 1)^r = (2^{\alpha-1} - 1).$$

Comparing the left-hand side and right-hand side, we get $r = 1$ and $\alpha - 1 = 2s$.
Hence $\alpha + \beta = 2s + 1 + 1 = 2(s + 1)$ and so $y = 2$. This implies that

$$F_k = 2^{\beta-1}(2^{\alpha-\beta} + 1) = 2^{2s} + 1,$$

is solvable only when $(s, F_k) = (1, 5)$. Consequently, $(s, x, y, F_k) = (1, 2, 2, 5)$ is a solution of (3.1). □

Theorem 3.2. *If L_k is a Lucas number, then the solutions of the Diophantine equation*

$$(2^{2s} - 1)^x + (2^{s+1})^y = L_k^2, \tag{3.2}$$

are $(s, x, y, L_k) \in \{(1, 1, 0, 2), (2, 0, 1, 3), (2, 1, 0, 4)\}$, where s is a positive integer and x, y, k are non-negative integers.

Proof. We consider the following exclusive cases for all non-negative integers x, y and k:
Case 1. If $k = 0$, then we have $L_0 = 2$, which implies that $(2^{2s} - 1)^x + (2^{s+1})^y = (2)^2$ is solvable only for $x = 1$ and $y = 0$. Hence $(s, x, y, L_k) = (1, 1, 0, 2)$ is a solution.
Case 2. If $x = 0$ in equation (3.2), then $1 + (2^{s+1})^y = L_k^2$ by Theorem 2.1, is solvable only for $(s, x, y, L_k) = (2, 0, 1, 3)$ and is a solution.
Case 3. If $y = 0$, then $(2^{2s} - 1)^x + 1 = L_k^2$ is solvable only when $(s, x, y, L_k) \in \{(1, 1, 0, 2), (2, 1, 0, 4)\}$.
Case 4. If $x = y = 0$, then we have $1 + 1 = L_k^2$ which is impossible.
Case 5. If $x = y = k = 0$, then we get $1 + 1 = 4$ which is impossible.
For the remaining possibilities, we assume that x, y, k are positive integers.
Case 6. If L_k be an even number, then $k = 3j$, for each $j \in \mathbb{Z}^+$. By Lemma 2.3, we have $L_k^2 \equiv 0 \pmod 4$ where $k = 3j$, for each $j \in \mathbb{Z}^+$. According to Lemmas 2.4 and 2.5, we have $(2^{2s} - 1)^x + (2^{s+1})^y \equiv 1$ or $3 \pmod 4$ for each positive integer y and $x = 2r$ or $2r + 1$ for each $r \in \mathbb{Z}^+$ which is impossible by our assumption.
Case 7. If L_k be an odd number, then $k = 3j - 2$ or $3j - 1$, for each $j \in \mathbb{Z}^+$. By Lemma 2.2, we have $L_k \equiv 1$ or $3 \pmod 4$ where $k = 3j - 2$ or $3j - 1$ for each $j \in \mathbb{Z}^+$ and by Lemma 2.3, we get $L_k^2 \equiv 1 \pmod 4$ where $k =$

$3j-2$ or $3j-1$ for each $j \in \mathbb{Z}^+$.
Also, by Lemmas 2.4 and 2.5, we have

$$(2^{2s}-1)^x + (2^{s+1})^y \equiv 1 \pmod 4,$$

for each positive integer y and $x = 2r$ and for each $r \in \mathbb{Z}^+$. Thus

$$(2^{2s}-1)^{2r} + (2^{s+1})^y = L_k^2,$$

$$(2^{s+1})^y = (L_k + (2^{2s}-1)^r)(L_k - (2^{2s}-1)^r),$$

there exist non-negative integers α and β where $\alpha+\beta = (s+1)y$ and $\alpha > \beta$, such that $2^\alpha = L_k + (2^{2s}-1)^r$ and $2^\beta = L_k - (2^{2s}-1)^r$. We get

$$2(2^{2s}-1)^r = 2^\beta(2^{\alpha-\beta}-1),$$

which implies that $2^\beta = 2$. Hence $\beta = 1$. Thus

$$(2^{2s}-1)^r = (2^{\alpha-1}-1).$$

Comparing the left-hand side and right-hand side, we get $r = 1$ and $\alpha - 1 = 2s$.
Hence $\alpha + \beta = 2s + 1 + 1 = 2(s+1)$. So $y = 2$. This shows that

$$L_k = 2^{\beta-1}(2^{\alpha-\beta}+1) = 2^{2s}+1,$$

is not solvable for any positive integers s and k. □

Conclusion

In this study, we discovered all solutions of the exponential Diophantine equations $(2^{2s}-1)^x + (2^{s+1})^y = F_k^2$ and $(2^{2s}-1)^x + (2^{s+1})^y = L_k^2$, where s is a positive number, F_k is a Fibonacci number, L_k is a Lucas number and x, y, k are non-negative integers. This should help readers try to solve exponential Diophantine equations for other special sequence numbers.

Acknowledgment. The authors would like to express their sincere gratitude to the referees for their careful reading of the manuscript and for their many useful comments and remarks that improved the quality of the paper.

References

[1] A. Suvarnamani, Solutions of the Diophantine equation $2^x + p^y = z^2$, Int. J. Math. Sci. Appl., **1,** no. 3, (2011), 1415–1419.

[2] Amr Elshahed, Hailiza Kamarulhaili, On the Diophantine Equation $(4^n)^x - p^y = z^2$, WSEAS Transactions on Mathematics, **19,** (2020), 349–352.

[3] D. Acu, On the Diophantine equation $2^x + 5^y = z^2$, General Mathematics, **15,** (2007), 145–148.

[4] B. Sroysang, On the Diophantine equation $3^x + 5^y = z^2$, Int. J. Pure and Appl. Math., **81,**(2011), 605–608.

[5] J. F. T. Rabago, On the Diophantine equation $2^x + 17^y = z^2$, Journal of the Indonesian Mathematical Society, **22,** no. 2, (2017), 177–182.

[6] P. B. Borah, M. Dutta, On the Diophantine equation $7^x + 32^y = z^2$ and Its generalization, Integers, **22,**(2022), A29.

[7] Apisit Pakapongpun, Bunthita Chattae, On the Diophantine equation $p^x + 7^y = z^2$, where p is prime and x, y, z are non-negative integers, International Journal of Mathematics and Computer Science, **17,** no. 4, (2022), 1535–1540.

[8] Suton Tadee, On the Diophantine Equation $n^x + 10^y = z^2$, WSEAS Transactions on Mathematics. **22,** (2023), 150–153.

[9] J. J. Bravo, F. Luca, On the Diophantine Equation $F_n + F_m = 2^a$, Quaestiones Mathematicae, **39,** no. 3, (2016), 391–400.

[10] D. Marques, A. Togbé, Fibonacci and Lucas numbers of the form $2^a + 3^b + 5^c$, Proc. Japan Acad., **89,** no. 3, (2013), 47–50.

[11] Y. Qu, J. Zeng, Pell and Pell-Lucas Numbers of the Form $-2^a - 3^b + 5^c$, Czechoslovak Mathematical Journal, **70,** no. 1, (2020), 281–289.

[12] Pagdame Tiebekabe, Ismail Diouf, On solutions of the Diophantine equation $L_n + L_m = 3^a$, Malaya Journal of Matematik, **9,** no. 4, (2021), 228–238.

[13] R. J. S. Mina, J. B. Bacani, On the Solutions of the Diophantine Equation $p^x + (p+4k)^y = z^2$ for Prime Pairs p and $p+4k$, European Journal of Pure and Applied Mathematics, **14,** no. 2, (2021), 471–479.

[14] S. Tadee, A. Siraworakun, Non-Existence of Positive Integer Solutions of the Diophantine Equation $p^x + (p+2q)^y = z^2$, where p, q and $p+2q$ are Prime Numbers, European Journal of Pure and Applied Mathematics, **16,** no. 2, (2023), 724–735.

[15] P. Mihăilescu, Primary cyclotomic units and a proof of Catalan's conjecture, *Journal für die Reine und Angewandte Mathematik*, **572**, (2004), 167–195.