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Matrix Variate Generalized Inverted Beta Distribution

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Abstract

The beta (type 1) and inverted beta distributions are of vital importance in science, engineering and economics. In this article, we introduce a matrix variate generalization of the inverted beta distribution. By using linear and quadratic transformations, we obtain several matrix variate distributions including matrix variate generalized beta type 1 distribution.

1 Introduction

A random variable X is said to have a beta type 2 or inverted beta distribution with shape parameters $\alpha > 0, \beta > 0$ and scale parameter $\sigma > 0$, denoted as $X \sim \mathrm{B2}(\alpha, \beta; \sigma)$ if its probability density function (pdf) is given by

$$\frac{x^{\alpha-1}(1+\sigma^{-1}x)^{-(\alpha+\beta)}}{\sigma^{\alpha}B(\alpha,\beta)}, \quad x > 0, \tag{1}$$

and $B(\alpha, \beta)$ is the usual beta function. The inverted beta or beta type 2 distribution is the most familiar statistical distribution in finance, economics and related areas. The growing applications of beta type 2 distribution have necessitated the need for more variations of this distribution. Several generalizations of this distribution are available

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in the literature. Following Nadarajah and Kotz [14], we define a generalization of the beta type 2 distribution by the pdf

$$K(\alpha,\beta,\mu,\nu,\sigma,\phi)x^{\alpha+\mu-1}(1+\sigma^{-1}x)^{-(\alpha+\beta)}(1+\phi^{-1}x)^{-(\mu+\nu)}, \quad x>0, \tag{2}$$

where $\alpha>0, \beta>0, \mu>0, \nu>0, \sigma>0$, $\phi>0$ and $K(\alpha,\beta,\mu,\nu,\sigma,\phi)$ denotes the normalizing constant. For $\sigma=\phi$, the density in (2) reduces to a beta type 2 density. Like the beta type 2 pdf, this pdf is unimodal and has more parameters than the usual beta type 2 distribution. Several properties of this distribution have been studied by Nadarajah and Kotz [14].

In this article, we give a matrix variate generalization of (2) and study its properties. Gupta and Nagar [6] gave a matrix variate generalization of (1). Several other matrix variate generalizations of beta distribution were given by Gupta and Nagar [6, 7, 8, 9, 10], Nagar and Gupta [15], Nagar, Arashi and Nadarajah [16], Nagar, Roldán-Correa and Gupta [18], and Nagar, Roldán-Correa and Nadarajah [19].

This paper is divided into four sections. In Section 2, we deal with some well known definitions and results on matrix algebra, zonal polynomials, invariant polynomials and special functions of matrix argument. In Section 3, we give the definition of the matrix variate generalized beta type 2 distribution and we devote Section 4 to the matrix variate generalized beta type 1 distribution.

2 Some Known Definitions and Results

We begin with a brief review of some definitions and notations. We adhere to the standard notations (cf. Gupta and Nagar [6], Muirhead [13]). Let $\boldsymbol{A}=(a_{ij})$ be an $m\times m$ matrix. Then, $\boldsymbol{A}'$ denotes the transpose of $\boldsymbol{A}$; $\operatorname{tr}(\boldsymbol{A})=a_{11}+\cdots+a_{mm}$; $\operatorname{etr}(\boldsymbol{A})=\exp(\operatorname{tr}(\boldsymbol{A}))$; $\det(\boldsymbol{A})=$ determinant of $\boldsymbol{A}$; norm of $\boldsymbol{A}=\|\boldsymbol{A}\|=$ maximum of absolute values of eigenvalues of $\boldsymbol{A}$; the identity matrix is denoted by $\boldsymbol{I}_m$; the null matrix is denoted by $\boldsymbol{O}$; $\boldsymbol{A}>\boldsymbol{O}$ means that $\boldsymbol{A}$ is symmetric positive definite (SPD); $\boldsymbol{A}^{1/2}$ denotes the unique SPD square root of $\boldsymbol{A}$ and $\boldsymbol{O}<\boldsymbol{A}<\boldsymbol{I}_m$ means that the matrices $\boldsymbol{A}$ and $\boldsymbol{I}_m-\boldsymbol{A}$ are SPD.

The multivariate gamma function which frequently occurs in multivariate statistical analysis is defined by $\Gamma_m(a)=\pi^{m(m-1)/4}\prod_{i=1}^m\Gamma\left(a-\frac{i-1}{2}\right)$, $\operatorname{Re}(a)>(m-1)/2$. We denote by $C_\kappa(\boldsymbol{X})$ the zonal polynomial of $m\times m$ complex symmetric matrix $\boldsymbol{X}$ corresponding to the ordered partition $\kappa=(k_1,\ldots,k_m), k_1\geq\cdots\geq k_m\geq 0$, $k_1+\cdots+k_m=k$ and $\sum_{\kappa\vdash k}$ denotes the summation over all partitions κ. The generalized hypergeometric coefficient $(a)_\kappa$ is defined by $(a)_\kappa=\prod_{i=1}^m\left(a-\frac{i-1}{2}\right)_{k_i}$, where $(a)_r=a(a+1)\cdots(a+r-1), r=1,2,\ldots$ with $(a)_0=1$. Moreover, we define $\Gamma_m(a,\kappa)$ as $\Gamma_m(a,\kappa)=(a)_\kappa\Gamma_m(a)$. Also, note that

$\Gamma_m(a,0)=\Gamma_m(a)$. Series expansions for ${}_1F_0(a;\boldsymbol{X})$ and ${}_2F_1(a,b;c;\boldsymbol{X})$ are given by

$$ {}_1F_0(a;\boldsymbol{X})=\sum_{k=0}^{\infty}\sum_{\kappa\vdash k}\frac{(a)_\kappa C_\kappa(\boldsymbol{X})}{k!}=\det(\boldsymbol{I}_m-\boldsymbol{X})^{-a},\ \|\boldsymbol{X}\|<1 $$

and

$$ {}_2F_1(a,b;c;\boldsymbol{X})=\sum_{k=0}^{\infty}\sum_{\kappa\vdash k}\frac{(a)_\kappa(b)_\kappa}{(c)_\kappa}\frac{C_\kappa(\boldsymbol{X})}{k!},\ \|\boldsymbol{X}\|<1. $$

The integral representation of the Gauss hypergeometric function ${}_2F_1$ is given by

$$ {}_2F_1(a,b;c;\boldsymbol{X})=\frac{\Gamma_m(c)}{\Gamma_m(a)\Gamma_m(c-a)}\int_{\boldsymbol{O}}^{\boldsymbol{I}_m}\frac{\det(\boldsymbol{R})^{a-(m+1)/2}\det(\boldsymbol{I}_m-\boldsymbol{R})^{c-a-(m+1)/2}}{\det(\boldsymbol{I}_m-\boldsymbol{X}\boldsymbol{R})^{b}}\,\mathrm{d}\boldsymbol{R}, \tag{3} $$

where $\mathrm{Re}(a)>(m-1)/2$ and $\mathrm{Re}(c-a)>(m-1)/2$. The Pfaff transformation formula for ${}_2F_1(a,b;c;\boldsymbol{X})$ is given by

$$ {}_2F_1(a,b;c;\boldsymbol{X})=\det(I_m-\boldsymbol{X})^{-b}\,{}_2F_1(c-a,b;c;-\boldsymbol{X}(I_m-\boldsymbol{X})^{-1}). \tag{4} $$

For properties and further results on these functions, we refer the reader to the works of Herz [11], Constantine [2], James [12], and Gupta and Nagar [6].

We denote by $C_\phi^{\kappa,\lambda}(\boldsymbol{X},\boldsymbol{Y})$ the invariant polynomial of $m\times m$ complex symmetric matrix arguments $\boldsymbol{X}$ and $\boldsymbol{Y}$ which is invariant under the transformation $(\boldsymbol{X},\boldsymbol{Y})\rightarrow(\boldsymbol{H}\boldsymbol{X}\boldsymbol{H}',\ \boldsymbol{H}\boldsymbol{Y}\boldsymbol{H}')$, $\boldsymbol{H}\in O(m)$, where $O(m)$ is the orthogonal group of $m\times m$ orthonormal matrices. For properties and applications of invariant polynomials, we refer the reader to the works of Davis [3, 4], Chikuse [1], Díaz-García [5], Nagar and Gupta [15], and Nagar and Nadarajah [17]. Let κ, λ and ϕ be ordered partitions of the non-negative integers k, ℓ, and $f=k+\ell$, respectively into not more than m parts. Then $C_\kappa^{\kappa,0}(\boldsymbol{X},\boldsymbol{Y})\equiv C_\kappa(\boldsymbol{X})$, $C_\lambda^{0,\lambda}(\boldsymbol{X},\boldsymbol{Y})\equiv C_\lambda(\boldsymbol{Y})$, $C_\phi^{\kappa,\lambda}(\boldsymbol{X},\boldsymbol{X})=\theta_\phi^{\kappa,\lambda}C_\phi(\boldsymbol{X})$,

$$ \theta_\phi^{\kappa,\lambda}=\frac{C_\phi^{\kappa,\lambda}(\boldsymbol{I}_m,\boldsymbol{I}_m)}{C_\phi(\boldsymbol{I}_m)},\ C_\phi^{\kappa,\lambda}(\boldsymbol{X},\boldsymbol{I}_m)=\theta_\phi^{\kappa,\lambda}\frac{C_\phi(\boldsymbol{I}_m)C_\kappa(\boldsymbol{X})}{C_\kappa(\boldsymbol{I}_m)},\ C_\phi^{\kappa,\lambda}(\boldsymbol{I}_m,\boldsymbol{Y})=\theta_\phi^{\kappa,\lambda}\frac{C_\phi(\boldsymbol{I}_m)C_\lambda(\boldsymbol{X})}{C_\lambda(\boldsymbol{I}_m)}, $$

$$ C_\kappa(\boldsymbol{X})C_\lambda(\boldsymbol{Y})=\sum_{\phi\in\kappa\cdot\lambda}\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(\boldsymbol{X},\boldsymbol{Y}), $$

where $\phi\in\kappa\cdot\lambda$ signifies that irreducible representation of $Gl(m,R)$ (group of $m\times m$ real non-singular matrices) indexed by 2ϕ, occurs in the decomposition of the Kronecker product $2\kappa\otimes2\lambda$ of the irreducible representations indexed by 2κ and 2λ (Davis [3]). Further, for symmetric matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of order m,

$$ \int_{\boldsymbol{O}}^{\boldsymbol{I}_m}\det(\boldsymbol{R})^{t-(m+1)/2}\det(I_m-\boldsymbol{R})^{u-(m+1)/2}C_\phi^{\kappa,\lambda}(\boldsymbol{A}\boldsymbol{R},\boldsymbol{B}\boldsymbol{R})\,\mathrm{d}\boldsymbol{R} $$

$$= \frac{\Gamma_m(t,\phi)\Gamma_m(u)}{\Gamma_m(t+u,\phi)} C_\phi^{\kappa,\lambda}(\boldsymbol{A},\boldsymbol{B}), \tag{5}$$

$$\int_{\boldsymbol{O}}^{\boldsymbol{I}_m} \det(\boldsymbol{R})^{t-(m+1)/2} \det(\boldsymbol{I}_m-\boldsymbol{R})^{u-(m+1)/2} C_\phi^{\kappa,\lambda}(\boldsymbol{A}\boldsymbol{R},\boldsymbol{B}(\boldsymbol{I}_m-\boldsymbol{R}))\,\mathrm{d}\boldsymbol{R}$$
$$= \frac{\Gamma_m(t,\kappa)\Gamma_m(u,\lambda)}{\Gamma_m(t+u,\phi)} C_\phi^{\kappa,\lambda}(\boldsymbol{A},\boldsymbol{B}). \tag{6}$$

The matrix variate beta and Gauss hypergeometric distributions are given as follows (Gupta and Nagar [6, 10]).

Definition 2.1. *An $m\times m$ random SPD matrix $\boldsymbol{U}$ is said to have a matrix variate beta type 1 distribution with parameters $(\alpha,\beta,\boldsymbol{\Omega})$, denoted as $\boldsymbol{U}\sim \mathrm{B1}(m,\alpha,\beta;\boldsymbol{\Omega})$, if its pdf is given by*

$$\frac{\Gamma_m(\alpha+\beta)\det(\boldsymbol{U})^{\alpha-(m+1)/2}\det(\boldsymbol{\Omega}-\boldsymbol{U})^{\beta-(m+1)/2}}{\det(\boldsymbol{\Omega})^{\alpha+\beta}\Gamma_m(\alpha)\Gamma_m(\beta)},\ \boldsymbol{O}<\boldsymbol{U}<\boldsymbol{\Omega},$$

where $\alpha>(m-1)/2$, $\beta>(m-1)/2$ and $\boldsymbol{\Omega}>\boldsymbol{O}$.

For $\boldsymbol{\Omega}=\boldsymbol{I}_m$, the above definition produces the standard beta type 1 distribution denoted by $\boldsymbol{U}\sim \mathrm{B1}(m,\alpha,\beta)$. Further, $\boldsymbol{\Omega}^{-1/2}\boldsymbol{U}\boldsymbol{\Omega}^{-1/2}\sim \mathrm{B1}(m,\alpha,\beta)$.

Definition 2.2. *An $m\times m$ random SPD matrix $\boldsymbol{V}$ is said to have a matrix variate beta type 2 or a matrix variate inverted beta distribution with parameters $(\alpha,\beta,\boldsymbol{\Omega})$, denoted as $\boldsymbol{V}\sim \mathrm{B2}(m,\alpha,\beta;\boldsymbol{\Omega})$, if its pdf is given by*

$$\frac{\Gamma_m(\alpha+\beta)\det(\boldsymbol{V})^{\alpha-(m+1)/2}\det(\boldsymbol{\Omega}+\boldsymbol{V})^{-(\alpha+\beta)}}{\det(\boldsymbol{\Omega})^{-\beta}\Gamma_m(\alpha)\Gamma_m(\beta)},\ \boldsymbol{V}>\boldsymbol{O},$$

where $\alpha>(m-1)/2$, $\beta>(m-1)/2$ and $\boldsymbol{\Omega}>\boldsymbol{O}$.

When $\boldsymbol{\Omega}=\boldsymbol{I}_m$, the above definition yields the standard beta type 2 distribution; that is, $\boldsymbol{V}\sim \mathrm{B2}(m,\alpha,\beta)$. Further, if $\boldsymbol{V}\sim \mathrm{B2}(m,\alpha,\beta;\boldsymbol{\Omega})$, then $\boldsymbol{\Omega}^{-1/2}\boldsymbol{V}\boldsymbol{\Omega}^{-1/2}\sim \mathrm{B2}(m,\alpha,\beta)$.

Definition 2.3. *An $m\times m$ random SPD matrix $\boldsymbol{X}$ is said to have a matrix variate Gauss hypergeometric distribution with parameters $(\alpha,\beta,\gamma,\boldsymbol{\Xi})$, denoted by $\boldsymbol{X}\sim \mathrm{GH}(m,\alpha,\beta,\gamma;\boldsymbol{\Xi})$, if its pdf is given by*

$$\frac{\Gamma_m(\alpha+\beta)}{\Gamma_m(\alpha)\Gamma_m(\beta)\,{}_2F_1(\alpha,\gamma;\alpha+\beta;-\boldsymbol{\Xi})}\frac{\det(\boldsymbol{X})^{\alpha-(m+1)/2}\det(\boldsymbol{I}_m-\boldsymbol{X})^{\beta-(m+1)/2}}{\det(\boldsymbol{I}_m+\boldsymbol{\Xi}\boldsymbol{X})^{\gamma}},\ \boldsymbol{O}<\boldsymbol{X}<\boldsymbol{I}_m,$$

where $\alpha>(m-1)/2$, $\beta>(m-1)/2$, $-\infty<\gamma<\infty$ and $\boldsymbol{I}_m+\boldsymbol{\Xi}>\boldsymbol{O}$.

Here, we note that if $\gamma = 0$ or $\boldsymbol{\Xi} = \boldsymbol{O}$, then $\boldsymbol{X} \sim \mathrm{B1}(m, \alpha, \beta)$.

Lemma 2.4. *For $a_i > (m-1)/2$, $b_i > (m-1)/2$, $i = 1, 2$, $\boldsymbol{B} > \boldsymbol{O}$ and $\boldsymbol{C} > \boldsymbol{O}$,*

$$\begin{aligned}
&\int_{\boldsymbol{X}>\boldsymbol{O}} \det(\boldsymbol{X})^{a_1+a_2-(m+1)/2} \det(\boldsymbol{I}_m + \boldsymbol{B}^{-1}\boldsymbol{X})^{-(a_1+b_1)} \det(\boldsymbol{I}_m + \boldsymbol{C}^{-1}\boldsymbol{X})^{-(a_2+b_2)} \,\mathrm{d}\boldsymbol{X} \\
&= \det(\boldsymbol{B})^{a_1-b_2} \det(\boldsymbol{C})^{a_2+b_2} \frac{\Gamma_m(a_1+a_2)\Gamma_m(b_1+b_2)}{\Gamma_m(a_1+a_2+b_1+b_2)} \\
&\quad \times {}_2F_1(b_1+b_2, a_2+b_2; a_1+a_2+b_1+b_2; \boldsymbol{I}_m - \boldsymbol{B}^{-1/2}\boldsymbol{C}\boldsymbol{B}^{-1/2}). \qquad (7)
\end{aligned}$$

Proof. Substituting $\boldsymbol{W} = (\boldsymbol{I}_m + \boldsymbol{B}^{-1/2}\boldsymbol{X}\boldsymbol{B}^{-1/2})^{-1}$ with the Jacobian $J(\boldsymbol{X} \to \boldsymbol{W}) = \det(\boldsymbol{B})^{(m+1)/2} \det(\boldsymbol{W})^{-(m+1)}$ in the above integral, one gets

$$\begin{aligned}
&\int_{\boldsymbol{X}>\boldsymbol{O}} \det(\boldsymbol{X})^{a_1+a_2-(m+1)/2} \det(\boldsymbol{I}_m + \boldsymbol{B}^{-1}\boldsymbol{X})^{-(a_1+b_1)} \det(\boldsymbol{I}_m + \boldsymbol{C}^{-1}\boldsymbol{X})^{-(a_2+b_2)} \,\mathrm{d}\boldsymbol{X} \\
&= \det(\boldsymbol{B})^{a_1-b_2} \det(\boldsymbol{C})^{a_2+b_2} \int_{\boldsymbol{O}}^{\boldsymbol{I}_m} \det(\boldsymbol{W})^{b_1+b_2-(m+1)/2} \det(\boldsymbol{I}_m - \boldsymbol{W})^{a_1+a_2-(m+1)/2} \\
&\quad \det(\boldsymbol{I}_m - (\boldsymbol{I}_m - \boldsymbol{B}^{-1/2}\boldsymbol{C}\boldsymbol{B}^{-1/2})\boldsymbol{W})^{-(a_2+b_2)} \,\mathrm{d}\boldsymbol{W}.
\end{aligned}$$

Now, the desired result is obtained by using (3). □

For a convergent series expansion of ${}_2F_1$, we require $||\boldsymbol{I}_m - \boldsymbol{B}^{-1/2}\boldsymbol{C}\boldsymbol{B}^{-1/2}|| < 1$. If $||\boldsymbol{I}_m - \boldsymbol{B}^{-1/2}\boldsymbol{C}\boldsymbol{B}^{-1/2}|| < 1$ is not met, then one can substitute $\boldsymbol{W} = (\boldsymbol{I}_m + \boldsymbol{C}^{-1/2}\boldsymbol{X}\boldsymbol{C}^{-1/2})^{-1}$ with the Jacobian $J(\boldsymbol{X} \to \boldsymbol{W}) = \det(\boldsymbol{C})^{(m+1)/2} \det(\boldsymbol{W})^{-(m+1)}$ in the integral in (7) to get

$$\begin{aligned}
&\int_{\boldsymbol{X}>\boldsymbol{O}} \det(\boldsymbol{X})^{a_1+a_2-(m+1)/2} \det(\boldsymbol{I}_m + \boldsymbol{B}^{-1}\boldsymbol{X})^{-(a_1+b_1)} \det(\boldsymbol{I}_m + \boldsymbol{C}^{-1}\boldsymbol{X})^{-(a_2+b_2)} \,\mathrm{d}\boldsymbol{X} \\
&= \det(\boldsymbol{C})^{a_2-b_1} \det(\boldsymbol{B})^{a_1+b_1} \frac{\Gamma_m(a_1+a_2)\Gamma_m(b_1+b_2)}{\Gamma_m(a_1+a_2+b_1+b_2)} \\
&\quad \times {}_2F_1(b_1+b_2, a_1+b_1; a_1+a_2+b_1+b_2; \boldsymbol{I}_m - \boldsymbol{C}^{-1/2}\boldsymbol{B}\boldsymbol{C}^{-1/2}), \qquad (8)
\end{aligned}$$

with the condition $||\boldsymbol{I}_m - \boldsymbol{C}^{-1/2}\boldsymbol{B}\boldsymbol{C}^{-1/2}|| < 1$. Note that (8) can also be obtained from (7) by using the Pfaff transformation given in (4).

3 Matrix variate generalized inverted beta distribution

The matrix variate generalization of (2) is given in the following definition.

Definition 3.1. *The SPD random matrix $\boldsymbol{X}$ of order m is said to have a matrix variate generalized inverted beta distribution, denoted by $\boldsymbol{X} \sim \mathrm{GB2}(m, \alpha, \beta, \mu, \nu; \boldsymbol{\Sigma}, \boldsymbol{\Phi})$, if its pdf is given by*

$$K(\alpha, \beta, \mu, \nu, \boldsymbol{\Sigma}, \boldsymbol{\Phi}) \frac{\det(\boldsymbol{X})^{\alpha+\mu-(m+1)/2}}{\det(\boldsymbol{I}_m + \boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{\alpha+\beta} \det(\boldsymbol{I}_m + \boldsymbol{\Phi}^{-1}\boldsymbol{X})^{\mu+\nu}}, \ \boldsymbol{X} > \boldsymbol{O},$$

where $\alpha > (m-1)/2, \beta > (m-1)/2, \mu > (m-1)/2, \nu > (m-1)/2, \boldsymbol{\Sigma} > \boldsymbol{O}, \boldsymbol{\Phi} > \boldsymbol{O}$ and $K(\alpha, \beta, \mu, \nu, \boldsymbol{\Sigma}, \boldsymbol{\Phi})$ is the normalizing constant.

By using (7) and (8), the normalizing constant is evaluated as

$$\begin{aligned}
[K(\alpha, \beta, \mu, \nu, \boldsymbol{\Sigma}, \boldsymbol{\Phi})]^{-1} &= \int_{\boldsymbol{X}>\boldsymbol{O}} \frac{\det(\boldsymbol{X})^{\alpha+\mu-(m+1)/2}}{\det(I_m + \boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{\alpha+\beta} \det(I_m + \boldsymbol{\Phi}^{-1}\boldsymbol{X})^{\mu+\nu}} \, \mathrm{d}\boldsymbol{X} \\
&= \det(\boldsymbol{\Sigma})^{\alpha-\nu} \det(\boldsymbol{\Phi})^{\mu+\nu} \frac{\Gamma_m(\alpha+\mu)\Gamma_m(\beta+\nu)}{\Gamma_m(\alpha+\beta+\mu+\nu)} \\
&\quad \times {}_2F_1(\beta+\nu, \mu+\nu; \alpha+\beta+\mu+\nu; \boldsymbol{I}_m - \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Phi}\boldsymbol{\Sigma}^{-1/2}),
\end{aligned}$$

where $||\boldsymbol{I}_m - \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Phi}\boldsymbol{\Sigma}^{-1/2}|| < 1$ and for $||\boldsymbol{I}_m - \boldsymbol{\Phi}^{-1/2}\boldsymbol{\Sigma}\boldsymbol{\Phi}^{-1/2}|| < 1$ the normalizing constant is given by

$$\begin{aligned}
[K(\alpha, \beta, \mu, \nu, \boldsymbol{\Sigma}, \boldsymbol{\Phi})]^{-1} &= \det(\boldsymbol{\Phi})^{\mu-\beta} \det(\boldsymbol{\Sigma})^{\alpha+\beta} \frac{\Gamma_m(\alpha+\mu)\Gamma_m(\beta+\nu)}{\Gamma_m(\alpha+\beta+\mu+\nu)} \\
&\quad \times {}_2F_1(\beta+\nu, \alpha+\beta; \alpha+\beta+\mu+\nu; \boldsymbol{I}_m - \boldsymbol{\Phi}^{-1/2}\boldsymbol{\Sigma}\boldsymbol{\Phi}^{-1/2}).
\end{aligned}$$

For $\boldsymbol{\Sigma} = \boldsymbol{\Phi} = \boldsymbol{\Omega}$, the matrix variate generalized inverted beta distribution slides to a matrix variate inverted beta distribution. That is, if $\boldsymbol{X} \sim \mathrm{GB2}(m, \alpha, \beta, \mu, \nu; \boldsymbol{\Sigma}, \boldsymbol{\Phi})$ and $\boldsymbol{\Sigma} = \boldsymbol{\Phi} = \boldsymbol{\Omega}$, then $\boldsymbol{X} \sim \mathrm{B2}(m, \alpha+\mu, \beta+\nu; \boldsymbol{\Omega})$. Several properties of this distribution are given in the remainder of this section.

Theorem 3.2. *If $\boldsymbol{X} \sim \mathrm{GB2}(m, \alpha, \beta, \mu, \nu; \boldsymbol{\Sigma}, \boldsymbol{\Phi})$, then $\boldsymbol{X}^{-1} \sim \mathrm{GB2}(m, \beta, \alpha, \nu, \mu; \boldsymbol{\Sigma}^{-1}, \boldsymbol{\Phi}^{-1})$.*

Proof. Transformaing $\boldsymbol{W} = \boldsymbol{X}^{-1}$ with the Jacobian $J(\boldsymbol{X} \to \boldsymbol{W}) = \det(\boldsymbol{W})^{-(m+1)}$ in the pdf of $\boldsymbol{X}$, we get the result. □

Theorem 3.3. *If $\boldsymbol{X} \sim \mathrm{GB2}(m, \alpha, \beta, \mu, \nu; \boldsymbol{\Sigma}, \boldsymbol{\Phi})$, then*

$$\boldsymbol{A}^{1/2}\boldsymbol{X}\boldsymbol{A}^{1/2} \sim \mathrm{GB2}(m, \alpha, \beta, \mu, \nu; \boldsymbol{A}^{1/2}\boldsymbol{\Sigma}\boldsymbol{A}^{1/2}, \boldsymbol{A}^{1/2}\boldsymbol{\Phi}\boldsymbol{A}^{1/2}).$$

Proof. Making the transformation $\boldsymbol{W} = \boldsymbol{A}^{1/2}\boldsymbol{X}\boldsymbol{A}^{1/2}$ with the Jacobian $J(\boldsymbol{X} \to \boldsymbol{W}) = \det(\boldsymbol{A})^{-(m+1)/2}$ in the pdf of $\boldsymbol{X}$, we get the result. □

Corollary 3.4. *If* $\boldsymbol{X} \sim \mathrm{GB2}(m, \alpha, \beta, \mu, \nu; \boldsymbol{\Sigma}, \boldsymbol{\Phi})$*, then*

$$\boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}\boldsymbol{\Sigma}^{-1/2} \sim \mathrm{GB2}(m, \alpha, \beta, \mu, \nu; \boldsymbol{I}_m, \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Phi}\boldsymbol{\Sigma}^{-1/2}).$$

Corollary 3.5. *If* $\boldsymbol{X} \sim \mathrm{GB2}(m, \alpha, \beta, \mu, \nu; \boldsymbol{\Sigma}, \boldsymbol{\Phi})$*, then*

$$\boldsymbol{\Phi}^{-1/2}\boldsymbol{X}\boldsymbol{\Phi}^{-1/2} \sim \mathrm{GB2}(m, \alpha, \beta, \mu, \nu; \boldsymbol{\Phi}^{-1/2}\boldsymbol{\Sigma}\boldsymbol{\Phi}^{-1/2}, \boldsymbol{I}_m).$$

The h-th moment of $\det(\boldsymbol{X})$ is derived as

$$\begin{aligned} E[\det(\boldsymbol{X})^h] &= K(\alpha, \beta, \mu, \nu, \boldsymbol{\Sigma}, \boldsymbol{\Phi}) \int_{\boldsymbol{X}>\boldsymbol{O}} \frac{\det(\boldsymbol{X})^{\alpha+\mu+h-(m+1)/2}}{\det(I_m + \boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{\alpha+\beta} \det(I_m + \boldsymbol{\Phi}^{-1}\boldsymbol{X})^{\mu+\nu}} \,\mathrm{d}\boldsymbol{X} \\ &= \frac{K(\alpha, \beta, \mu, \nu, \boldsymbol{\Sigma}, \boldsymbol{\Phi})}{K(\alpha+h, \beta-h, \mu, \nu, \boldsymbol{\Sigma}, \boldsymbol{\Phi})}. \end{aligned}$$

Now, substituting appropriately in the above expression, we get

$$\begin{aligned} E[\det(\boldsymbol{X})^h] &= \det(\boldsymbol{\Sigma})^h \frac{\Gamma_m(\alpha+\mu+h)\Gamma_m(\beta+\nu-h)}{\Gamma_m(\alpha+\mu)\Gamma_m(\beta+\nu)} \\ &\quad \times \frac{{}_2F_1(\beta+\nu-h, \mu+\nu; \alpha+\beta+\mu+\nu; \boldsymbol{I}_m - \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Phi}\boldsymbol{\Sigma}^{-1/2})}{{}_2F_1(\beta+\nu, \mu+\nu; \alpha+\beta+\mu+\nu; \boldsymbol{I}_m - \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Phi}\boldsymbol{\Sigma}^{-1/2})}, \end{aligned}$$

for $||\boldsymbol{I}_m - \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Phi}\boldsymbol{\Sigma}^{-1/2}|| < 1$ and for $||\boldsymbol{I}_m - \boldsymbol{\Phi}^{-1/2}\boldsymbol{\Sigma}\boldsymbol{\Phi}^{-1/2}|| < 1$,

$$\begin{aligned} E[\det(\boldsymbol{X})^h] &= \det(\boldsymbol{\Phi})^h \frac{\Gamma_m(\alpha+\mu+h)\Gamma_m(\beta+\nu-h)}{\Gamma_m(\alpha+\mu)\Gamma_m(\beta+\nu)} \\ &\quad \times \frac{{}_2F_1(\beta+\nu-h, \alpha+\beta; \alpha+\beta+\mu+\nu; \boldsymbol{I}_m - \boldsymbol{\Phi}^{-1/2}\boldsymbol{\Sigma}\boldsymbol{\Phi}^{-1/2})}{{}_2F_1(\beta+\nu, \alpha+\beta; \alpha+\beta+\mu+\nu; \boldsymbol{I}_m - \boldsymbol{\Phi}^{-1/2}\boldsymbol{\Sigma}\boldsymbol{\Phi}^{-1/2})}. \end{aligned}$$

4 Matrix variate generalized beta distribution

It is well known that (Gupta and Nagar [6]) if $\boldsymbol{V} \sim \mathrm{B2}(m, \alpha, \beta)$, then $(\boldsymbol{I}_m + \boldsymbol{V})^{-1} \sim \mathrm{B1}(m, \beta, \alpha)$ and $(\boldsymbol{I}_m + \boldsymbol{V})^{-1}\boldsymbol{V} \sim \mathrm{B1}(m, \alpha, \beta)$. In this section, we derive a similar result for the matrix variate generalized inverted beta distribution.

Theorem 4.1. *If* $\boldsymbol{Z} = (\boldsymbol{I}_m + \boldsymbol{A}^{1/2}\boldsymbol{X}\boldsymbol{A}^{1/2})^{-1}$*, then the pdf of* $\boldsymbol{Z}$ *is given by*

$$\begin{aligned} &K(\alpha, \beta, \mu, \nu, \boldsymbol{\Sigma}, \boldsymbol{\Phi}) \det(\boldsymbol{\Phi})^{\mu+\nu} \det(\boldsymbol{\Sigma})^{\alpha+\beta} \det(\boldsymbol{A})^{\beta+\nu} \\ &\times \frac{\det(\boldsymbol{Z})^{\beta+\nu-(m+1)/2} \det(\boldsymbol{I}_m - \boldsymbol{Z})^{\alpha+\mu-(m+1)/2}}{\det(\boldsymbol{I}_m - (\boldsymbol{I}_m - \boldsymbol{A}^{1/2}\boldsymbol{\Sigma}\boldsymbol{A}^{1/2})\boldsymbol{Z})^{\alpha+\beta} \det(\boldsymbol{I}_m - (\boldsymbol{I}_m - \boldsymbol{A}^{1/2}\boldsymbol{\Phi}\boldsymbol{A}^{1/2})\boldsymbol{Z})^{\mu+\nu}}, \ \boldsymbol{O} < \boldsymbol{Z} < \boldsymbol{I}_m. \end{aligned}$$

Proof. From Theorem 3.3, the density of $\boldsymbol{W} = \boldsymbol{A}^{1/2}\boldsymbol{X}\boldsymbol{A}^{1/2}$ is given by

$$K(\alpha,\beta,\mu,\nu,\boldsymbol{\Sigma}_1,\boldsymbol{\Phi}_1)\frac{\det(\boldsymbol{W})^{\alpha+\mu-(m+1)/2}}{\det(\boldsymbol{I}_m+\boldsymbol{\Sigma}_1^{-1}\boldsymbol{W})^{\alpha+\beta}\det(\boldsymbol{I}_m+\boldsymbol{\Phi}_1^{-1}\boldsymbol{W})^{\mu+\nu}},\ \boldsymbol{W}>\boldsymbol{O},$$

where $\boldsymbol{\Sigma}_1 = \boldsymbol{A}^{1/2}\boldsymbol{\Sigma}\boldsymbol{A}^{1/2}$ and $\boldsymbol{\Phi}_1 = \boldsymbol{A}^{1/2}\boldsymbol{\Phi}\boldsymbol{A}^{1/2}$. Now, transforming $\boldsymbol{Z} = (\boldsymbol{I}_m+\boldsymbol{W})^{-1}$ with the Jacobian $J(\boldsymbol{W}\to\boldsymbol{Z}) = \det(\boldsymbol{Z})^{-m-1}$ in the above pdf, the pdf of $\boldsymbol{Z}$ is obtained. □

Corollary 4.2. $\boldsymbol{Z} \sim \mathrm{GH}(m,\beta+\nu,\alpha+\mu,\alpha+\beta+\mu+\nu,-(\boldsymbol{I}_m-\boldsymbol{A}^{1/2}\boldsymbol{\Omega}\boldsymbol{A}^{1/2}))$ *if* $\boldsymbol{\Phi}=\boldsymbol{\Sigma}=\boldsymbol{\Omega}$.

Corollary 4.3. *If* $\boldsymbol{Z}_1 = (\boldsymbol{I}_m+\boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}\boldsymbol{\Sigma}^{-1/2})^{-1}$, *then* $\boldsymbol{Z}_1 \sim \mathrm{GH}(m,\beta+\nu,\alpha+\mu,\alpha+\beta, -(\boldsymbol{I}_m-\boldsymbol{\Phi}^{-1/2}\boldsymbol{\Sigma}\boldsymbol{\Phi}^{-1/2}))$. *Moreover, if* $\boldsymbol{\Phi}=\boldsymbol{\Sigma}$, *then* $\boldsymbol{Z}_1\sim \mathrm{B1}(m,\beta+\nu,\alpha+\mu)$.

Corollary 4.4. *If* $\boldsymbol{Z}_2 = (\boldsymbol{I}_m+\boldsymbol{\Phi}^{-1/2}\boldsymbol{X}\boldsymbol{\Phi}^{-1/2})^{-1}$, *then* $\boldsymbol{Z}_2 \sim \mathrm{GH}(m,\beta+\nu,\alpha+\mu,\mu+\nu, -(\boldsymbol{I}_m-\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Phi}\boldsymbol{\Sigma}^{-1/2}))$ *and if* $\boldsymbol{\Phi}=\boldsymbol{\Sigma}$, *then* $\boldsymbol{Z}_2\sim \mathrm{B1}(m,\beta+\nu,\alpha+\mu)$.

Corollary 4.5. *If* $\boldsymbol{Z} = (\boldsymbol{I}_m+\boldsymbol{X})^{-1}$, *then the pdf of* $\boldsymbol{Z}$ *is given by*

$$K(\alpha,\beta,\mu,\nu,\boldsymbol{\Sigma},\boldsymbol{\Phi})\det(\boldsymbol{\Phi})^{\mu+\nu}\det(\boldsymbol{\Sigma})^{\alpha+\beta}\frac{\det(\boldsymbol{Z})^{\beta+\nu-(m+1)/2}\det(\boldsymbol{I}_m-\boldsymbol{Z})^{\alpha+\mu-(m+1)/2}}{\det(\boldsymbol{I}_m-(\boldsymbol{I}_m-\boldsymbol{\Sigma})\boldsymbol{Z})^{\alpha+\beta}\det(\boldsymbol{I}_m-(\boldsymbol{I}_m-\boldsymbol{\Phi})\boldsymbol{Z})^{\mu+\nu}}.$$

Moreover, if $\boldsymbol{\Phi}=\boldsymbol{\Sigma}=\boldsymbol{\Omega}$, *then* $\boldsymbol{Z}\sim\mathrm{GH}(m,\beta+\nu,\alpha+\mu,\alpha+\beta+\mu+\nu,-(\boldsymbol{I}_m-\boldsymbol{\Omega}))$.

Corollary 4.6. *If* $\boldsymbol{R} = (\boldsymbol{I}_m+\boldsymbol{X})^{-1/2}\boldsymbol{X}(\boldsymbol{I}_m+\boldsymbol{X})^{-1/2}$, *then the pdf of* $\boldsymbol{R}$ *is given by*

$$K(\alpha,\beta,\mu,\nu,\boldsymbol{\Sigma},\boldsymbol{\Phi})\frac{\det(\boldsymbol{R})^{\alpha+\mu-(m+1)/2}\det(\boldsymbol{I}_m-\boldsymbol{R})^{\beta+\nu-(m+1)/2}}{\det(\boldsymbol{I}_m-(\boldsymbol{I}_m-\boldsymbol{\Sigma}^{-1})\boldsymbol{R})^{\alpha+\beta}\det(\boldsymbol{I}_m-(\boldsymbol{I}_m-\boldsymbol{\Phi}^{-1})\boldsymbol{R})^{\mu+\nu}},\ \boldsymbol{O}<\boldsymbol{R}<\boldsymbol{I}_m$$

and if $\boldsymbol{\Phi}=\boldsymbol{\Sigma}=\boldsymbol{\Omega}$, *then* $\boldsymbol{R}\sim\mathrm{GH}(m,\alpha+\mu,\beta+\nu,\alpha+\beta+\mu+\nu,-(\boldsymbol{I}_m-\boldsymbol{\Omega}^{-1}))$.

Using the density of $\boldsymbol{R}$ given above

$$E[\det(\boldsymbol{R})^h] = K(\alpha,\beta,\mu,\nu,\boldsymbol{\Sigma},\boldsymbol{\Phi}) \times\int_{\boldsymbol{O}}^{\boldsymbol{I}_m}\frac{\det(\boldsymbol{R})^{\alpha+\mu+h-(m+1)/2}\det(\boldsymbol{I}_m-\boldsymbol{R})^{\beta+\nu-(m+1)/2}}{\det(\boldsymbol{I}_m-(\boldsymbol{I}_m-\boldsymbol{\Sigma}^{-1})\boldsymbol{R})^{\alpha+\beta}\det(\boldsymbol{I}_m-(\boldsymbol{I}_m-\boldsymbol{\Phi}^{-1})\boldsymbol{R})^{\mu+\nu}}\mathrm{d}\boldsymbol{R}. \quad (9)$$

To evaluate this integral, we consider four different cases; namely, (i) $\|\boldsymbol{I}_m-\boldsymbol{\Sigma}^{-1}\|<1$, $\|\boldsymbol{I}_m-\boldsymbol{\Phi}^{-1}\|<1$, (ii) $\|\boldsymbol{I}_m-\boldsymbol{\Sigma}^{-1}\|>1$, $\|\boldsymbol{I}_m-\boldsymbol{\Phi}^{-1}\|<1$, (iii) $\|\boldsymbol{I}_m-\boldsymbol{\Sigma}^{-1}\|<1$, $\|\boldsymbol{I}_m-\boldsymbol{\Phi}^{-1}\|>1$, and (iv) $\|\boldsymbol{I}_m-\boldsymbol{\Sigma}^{-1}\|>1$, $\|\boldsymbol{I}_m-\boldsymbol{\Phi}^{-1}\|>1$. Expanding the denominator of the integrand in (9) for (i), (ii), (iii) and (iv), by using results on zonal and invariant polynomials given in Section 2, one gets

$$\det(\boldsymbol{I}_m-(\boldsymbol{I}_m-\boldsymbol{\Sigma}^{-1})\boldsymbol{R})^{-(\alpha+\beta)}\det(\boldsymbol{I}_m-(\boldsymbol{I}_m-\boldsymbol{\Phi}^{-1})\boldsymbol{R})^{-(\mu+\nu)}$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\kappa}\sum_{\lambda}\frac{(\alpha+\beta)_{\kappa}(\mu+\nu)_{\lambda}}{k!\,l!}\sum_{\phi\in\kappa\cdot\lambda}\theta_{\phi}^{\kappa,\lambda}C_{\phi}^{\kappa,\lambda}((\boldsymbol{I}_m-\boldsymbol{\Sigma}^{-1})\boldsymbol{R},(\boldsymbol{I}_m-\boldsymbol{\Phi}^{-1})\boldsymbol{R}), \quad (10)$$

$$\begin{aligned}&\det(\boldsymbol{I}_m-(\boldsymbol{I}_m-\boldsymbol{\Sigma}^{-1})\boldsymbol{R})^{-(\alpha+\beta)}\det(\boldsymbol{I}_m-(\boldsymbol{I}_m-\boldsymbol{\Phi}^{-1})\boldsymbol{R})^{-(\mu+\nu)}\\ &\quad=\det(\boldsymbol{\Sigma})^{\alpha+\beta}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\kappa}\sum_{\lambda}\frac{(\alpha+\beta)_{\kappa}(\mu+\nu)_{\lambda}}{k!\,l!}\\ &\qquad\times\sum_{\phi\in\kappa\cdot\lambda}\theta_{\phi}^{\kappa,\lambda}C_{\phi}^{\kappa,\lambda}((\boldsymbol{I}_m-\boldsymbol{\Sigma})(\boldsymbol{I}_m-\boldsymbol{R}),(\boldsymbol{I}_m-\boldsymbol{\Phi}^{-1})\boldsymbol{R}),\end{aligned} \quad (11)$$

$$\begin{aligned}&\det(\boldsymbol{I}_m-(\boldsymbol{I}_m-\boldsymbol{\Sigma}^{-1})\boldsymbol{R})^{-(\alpha+\beta)}\det(\boldsymbol{I}_m-(\boldsymbol{I}_m-\boldsymbol{\Phi}^{-1})\boldsymbol{R})^{-(\mu+\nu)}\\ &\quad=\det(\boldsymbol{\Phi})^{\mu+\nu}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\kappa}\sum_{\lambda}\frac{(\alpha+\beta)_{\kappa}(\mu+\nu)_{\lambda}}{k!\,l!}\\ &\qquad\times\sum_{\phi\in\kappa\cdot\lambda}\theta_{\phi}^{\kappa,\lambda}C_{\phi}^{\kappa,\lambda}((\boldsymbol{I}_m-\boldsymbol{\Sigma}^{-1})\boldsymbol{R},(\boldsymbol{I}_m-\boldsymbol{\Phi})(\boldsymbol{I}_m-\boldsymbol{R})),\end{aligned} \quad (12)$$

$$\begin{aligned}&\det(\boldsymbol{I}_m-(\boldsymbol{I}_m-\boldsymbol{\Sigma}^{-1})\boldsymbol{R})^{-(\alpha+\beta)}\det(\boldsymbol{I}_m-(\boldsymbol{I}_m-\boldsymbol{\Phi}^{-1})\boldsymbol{R})^{-(\mu+\nu)}\\ &\quad=\det(\boldsymbol{\Sigma})^{\alpha+\beta}\det(\boldsymbol{\Phi})^{\mu+\nu}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\kappa}\sum_{\lambda}\frac{(\alpha+\beta)_{\kappa}(\mu+\nu)_{\lambda}}{k!\,l!}\\ &\qquad\times\sum_{\phi\in\kappa\cdot\lambda}\theta_{\phi}^{\kappa,\lambda}C_{\phi}^{\kappa,\lambda}((\boldsymbol{I}_m-\boldsymbol{\Sigma})(\boldsymbol{I}_m-\boldsymbol{R}),(\boldsymbol{I}_m-\boldsymbol{\Phi})(\boldsymbol{I}_m-\boldsymbol{R})).\end{aligned} \quad (13)$$

Now, substituting appropriately from (10), (11), (12) and (13) into (9) and integrating out $\boldsymbol{R}$ by using (5) and (6), we get

$$\begin{aligned}E[\det(\boldsymbol{R})^h]&=K(\alpha,\beta,\mu,\nu,\boldsymbol{\Sigma},\boldsymbol{\Phi})\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\kappa}\sum_{\lambda}\frac{(\alpha+\beta)_{\kappa}(\mu+\nu)_{\lambda}}{k!\,l!}\\ &\quad\times\sum_{\phi\in\kappa\cdot\lambda}\frac{\Gamma_m(\alpha+\mu+h,\phi)\Gamma_m(\beta+\nu)}{\Gamma_m(\alpha+\beta+\mu+\nu+h,\phi)}\theta_{\phi}^{\kappa,\lambda}C_{\phi}^{\kappa,\lambda}(\boldsymbol{I}_m-\boldsymbol{\Sigma}^{-1},\boldsymbol{I}_m-\boldsymbol{\Phi}^{-1}),\end{aligned}$$

$$\begin{aligned}E[\det(\boldsymbol{R})^h]&=K(\alpha,\beta,\mu,\nu,\boldsymbol{\Sigma},\boldsymbol{\Phi})\det(\boldsymbol{\Sigma})^{\alpha+\beta}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\kappa}\sum_{\lambda}\frac{(\alpha+\beta)_{\kappa}(\mu+\nu)_{\lambda}}{k!\,l!}\\ &\quad\times\sum_{\phi\in\kappa\cdot\lambda}\frac{\Gamma_m(\alpha+\mu+h,\kappa)\Gamma_m(\beta+\nu,\lambda)}{\Gamma_m(\alpha+\beta+\mu+\nu+h,\phi)}\theta_{\phi}^{\kappa,\lambda}C_{\phi}^{\kappa,\lambda}(\boldsymbol{I}_m-\boldsymbol{\Sigma},\boldsymbol{I}_m-\boldsymbol{\Phi}^{-1}),\end{aligned}$$

$$E[\det(\boldsymbol{R})^h] = K(\alpha,\beta,\mu,\nu,\boldsymbol{\Sigma},\boldsymbol{\Phi})\det(\boldsymbol{\Phi})^{\mu+\nu}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\kappa}\sum_{\lambda}\frac{(\alpha+\beta)_\kappa(\mu+\nu)_\lambda}{k!\,l!}$$
$$\times\sum_{\phi\in\kappa\cdot\lambda}\frac{\Gamma_m(\alpha+\mu+h,\kappa)\Gamma_m(\beta+\nu,\lambda)}{\Gamma_m(\alpha+\beta+\mu+\nu+h,\phi)}\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(\boldsymbol{I}_m-\boldsymbol{\Sigma}^{-1},\boldsymbol{I}_m-\boldsymbol{\Phi}),$$

$$E[\det(\boldsymbol{R})^h] = K(\alpha,\beta,\mu,\nu,\boldsymbol{\Sigma},\boldsymbol{\Phi})\det(\boldsymbol{\Sigma})^{\alpha+\beta}\det(\boldsymbol{\Phi})^{\mu+\nu}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\kappa}\sum_{\lambda}\frac{(\alpha+\beta)_\kappa(\mu+\nu)_\lambda}{k!\,l!}$$
$$\times\sum_{\phi\in\kappa\cdot\lambda}\frac{\Gamma_m(\alpha+\mu+h)\Gamma_m(\beta+\nu,\phi)}{\Gamma_m(\alpha+\beta+\mu+\nu+h,\phi)}\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(\boldsymbol{I}_m-\boldsymbol{\Sigma},\boldsymbol{I}_m-\boldsymbol{\Phi}).$$

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