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Proposed Multi-Dimensional Algebra

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Abstract

In this paper, we establish a new eight-dimensional algebra callked KAH-Octo. Moreover, we study subalgebras aand give some properties of the algebra such as division, isomorphism, simple, semi-simple, Jordan, Malcev, among others. Furthermore, we give some applications to some interesting areas of mathematics such as cryptography.

1 Introduction

Many researchers have presented different types of multi-dimensional algebra and we will mention a number of them. In 2016, Yassein and Al-Saidi [1] proposed that hexadecnion algebra has 16 dimensions. In 2018, Yassein and Al-Saidi introduced bicartesian algebra has two dimensions [2]. In 2020,

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Yassein et al. [3] proposed carternion algebra which has four dimensions. In 2021, Abo-Alsood [4] and Yassein introduced bi-octonion algebra with basis $\{1, e_1\}$. Also in 2021, Shahhadi and Yassein [5] proposed tripternion algebra with basis $\{1, x, x^2\}$.

2 KAH-Octo Algebra

In this section, we propose a new multidimensional algebra over the field F, called KAH-Octo and denoted by KO, as follows:
$KO = \{x : x = \sum_{i=0}^{7} x_i\beta_i \,|\, x_0, \ldots, x_7 \in F, \beta_0 = 0\}$. Now, define the operators addition, multiplication, and scalar multiplication as follows:
Let $x, y \in KO$ and $a \in F$ such that

$$x = \sum_{i=0}^{7} x_i\beta_i, y = \sum_{i=0}^{7} y_i\beta_i.$$

Then

$$x + y = \sum_{i=0}^{7} x_i\beta_i + \sum_{i=0}^{7} y_i\beta_i = \sum_{i=0}^{7}(x_i + y_i)\beta_i,$$

$$x * y = \sum_{i=0}^{7}(x_iy_i)\beta_i,$$

and

$$a.x = a.\left(\sum_{i=0}^{7} x_i\beta_i\right) = \sum_{i=0}^{7} ax_i\beta_i,$$

respectively.
It is clear that $(KO, +, .)$ is a vector space with a basis
$\{\beta_0 = 1, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7\}$.

Proposition 2.1. *There are eight forms of subspaces of the vector space* KO.

Proof. Let $w_i \subseteq KO$, $i = \{0, \ldots, 7\}$
1) The first form is $w_1 = \{x_i\beta_i\}$:
Let $x, y \in w_1$ such that $x = x_i\beta_i$, $y = y_i\beta_i$.
i) $x + y = x_i\beta_i + y_i\beta_i = (x_i + y_i)\beta_i \in w_1$.
ii) $a.x = a.x_i\beta_i = (ax_i)\beta_i \in w_1$.
Therefore, w_1 is a subspace and the number of this form is eight.

2) The second form is $w_2 = \sum_{j=1}^{2} x_{ij}\beta_{ij}$.

Let x,$y \in w_2$ such that $x = \sum_{j=1}^{2} x_{ij}\beta_{ij}$, $\sum_{j=1}^{2} y_{ij}\beta_{ij}$.

i) $x + y = \sum_{j=1}^{2} x_{ij}\beta_{ij} + \sum_{j=1}^{2} y_{ij}\beta_{ij} = \sum_{j=1}^{2}(x_{ij} + y_{ij})\beta_{ij} \in w_2$.

ii) $a.x = a.\sum_{j=1}^{2} x_{ij}\beta_{ij} = \sum_{j=1}^{2}(ax_{ij})\beta_{ij} \in w_2$.

Therefore, w_2 is a subspace and the number of this form is 28.

The proofs for the following remaining cases are similar.

3) The third form is $w_3 = \sum_{j=1}^{3} x_{ij}\beta_{ij}$ and the number of this form is 56.

4) The fourth form is $w_4 = \sum_{j=1}^{4} x_{ij}\beta_{ij}$ and the number of this form is 70.

5) The fifth form is $w_5 = \sum_{j=1}^{5} x_{ij}\beta_{ij}$ and the number of this form is 56.

6) The sixth form is $w_6 = \sum_{j=1}^{6} x_{ij}\beta_{ij}$ and the number of this form is 28.

7) The seventh form is $w_7 = \sum_{j=1}^{7} x_{ij}\beta_{ij}$ and the number of this form is eight.

8) The eighth form is $w_8 = \sum_{j=1}^{8} x_{ij}\beta_{ij}$ and the number of this form is 1.

Proposition 2.2. *The vector space KO is a commutative algebra with multiplication* $*$.

Proof. Let x,y,$z \in KO$, $a \in F$ such that

$$x = \sum_{i=0}^{7} x_i\beta_i, y = \sum_{i=0}^{7} y_i\beta_i, z = \sum_{i=0}^{7} z_i\beta_i.$$

$$1) x * (y + z) = \sum_{i=0}^{7} x_i\beta_i * \left(\sum_{i=0}^{7} y_i\beta_i + \sum_{i=0}^{7} x_i\beta_i\right) = \sum_{i=0}^{7} x_i\beta_i * \sum_{i=0}^{7}(y_i + z_i)\beta_i$$

$$= \sum_{i=0}^{7} x_i(y_i+z_i)\beta_i = \sum_{i=0}^{7}(x_iy_i+x_iz_i)\beta_i = \sum_{i=0}^{7}(x_iy_i)\beta_i+\sum_{i=0}^{7}(x_iz_i)\beta_i = x*y+x*z.$$

2) Similarly, $(x + y) * z = x * z + y * z$.

$$3) a.(x*y) = a.\left(\sum_{i=0}^{7} x_i\beta_i * \sum_{i=0}^{7} y_i\beta_i\right) = a.\sum_{i=0}^{7}(x_iy_i)\beta_i = \sum_{i=0}^{7} a(x_iy_i)\beta_i = \sum_{i=0}^{7}((ax_i)y_i)\beta_i = (a.x)*y$$

and

$$(a.x)*y = \sum_{i=0}^{7}((ax_i)y_i)\beta_i = \sum_{i=0}^{7}((x_ia)y_i)\beta_i = \sum_{i=0}^{7}(x_i(ay_i))\beta_i = \sum_{i=0}^{7} x_i\beta_i*\sum_{i=0}^{7}(ay_i)\beta_i$$

$$= \sum_{i=0}^{7} x_i\beta_i * a.\sum_{i=0}^{7} y_i\beta_i = x * (ay).$$

Therefore, KO is an algebra.
4) $x*y=\sum_{i=0}^{7}(x_iy_i)\beta_i=\sum_{i=0}^{7}(y_ix)_i\beta_i=y*x$. Hence KO is commutative.

Remark 2.3. *Every subspace of KO is a subalgebra of the KO algebra with multiplication $*$.*

Corollary 2.4. *The Algebra KO is associative.*

Proof.

$$(x*y)*z=\left(\sum_{i=0}^{7}x_i\beta_i*\sum_{i=0}^{7}y_i\beta_i\right)*\sum_{i=0}^{7}z_i\beta_i=\sum_{i=0}^{7}(x_iy_i)\beta_i*\sum_{i=0}^{7}z_i\beta_i=\sum_{i=0}^{7}((x_iy_i)z_i)\beta_i$$

$$=\sum_{i=0}^{7}(x_i(y_iz_i))\beta_i=\left(\sum_{i=0}^{7}x_i\beta_i*\sum_{i=0}^{7}y_iz_i\beta_i\right)=\sum_{i=0}^{7}x_i\beta_i*\left(\sum_{i=0}^{7}y_i\beta_i*\sum_{i=0}^{7}z_i\beta_i\right)=x*(y*z).$$

Remark 2.5. *The identity element in KO algebra is $1+\sum_{i=1}^{7}\beta_i$, the multiplicative inverse of x is $x^{-1}=\sum_{i=0}^{7}x_i^{-1}\beta_i$ such that $x_i\neq 0\ \forall\, i=0,\ldots,7$, and KO algebra is not division because if $x=1+\beta_1$, $y=\beta_2+\beta_3$ then $x*y=0$.*

Proposition 2.6. *In the field R, $(KO,+,.,*)\cong(R^8,+,.,\circledast)$ such that $(x_0,\ldots,x_7)\circledast(y_0,\ldots,y_7)=(x_0y_0,\ldots,x_7y_7)$.*

Proof. Let x,$y\in KO$ and a,$b\in R$ such that

$$x=\sum_{i=0}^{7}x_i\beta_i,\, y=\sum_{i=0}^{7}y_i\beta_i.$$

Define $f:(KO,+,.,*)\to(R^8,+,.,\circledast)$ such that

$$f(\sum_{i=0}^{7}x_i\beta_i)=(x_0,\ldots,x_7).$$

Then

$$1)f(a.x+b.y)=f(a.\sum_{i=0}^{7}x_i\beta_i+b.\sum_{i=0}^{7}y_i\beta_i)=f(\sum_{i=0}^{7}(ax_i)\beta_i+\sum_{i=0}^{7}(by_i)\beta_i)$$

$$=f(\sum_{i=0}^{7}(ax_i+by_i)\beta_i)=(ax_0+by_0+ax_1+by_1+\cdots+ax_7+by_7)=(ax_0+ax_1+\cdots+ax_7)$$

$+$

$$(by_0+by_1+\cdots+by_7)=a(x_0+x_1+\cdots+x_7)+b(y_0+y_1+\cdots+y_7)=a.f(x)+b.f(y).$$

$$2) f(x * y) = f(\sum_{i=0}^{7} x_i\beta_i * \sum_{i=0}^{7} y_i\beta_i) = f(\sum_{i=0}^{7}(x_iy_i)\beta_i) = (x_0y_0, \ldots, x_7y_7)$$

$$= (x_0, \ldots, x_7) \circledast (y_0, \ldots, y_7) = f(\sum_{i=0}^{7} x_i\beta_i) \circledast f(\sum_{i=0}^{7} y_i\beta_i) = f(x) \circledast f(y).$$

$3) f = \{x \in KO \,|\, f(x) = 0\} = \{x \in KO \,|\, f(x) = (0,0,0,0,0,0,0)\} = \{x \in KO \,|\, (x_0, x_1, \ldots, x_7)$
$= (0,0,0,0,0,0,0)\} = \{x \in KO \,|\, x_0 = x_1 = \cdots = x_7 = 0\} = \{x \in KO \,|\, x = 0\}.$

4) Let $(x_0, \ldots, x_7) \in R^8$. Then

$$x = \sum_{i=0}^{7} x_i\beta_i \in KO$$

such that

$$f(x) = f(\sum_{i=0}^{7} x_i\beta_i) = (x_0, \ldots, x_7).$$

Therefore, f is onto.
Consequently, $(KO, +, ., *) \cong (R^8, +, ., \circledast)$.

Proposition 2.7. *1) Every subalgebra I of KO is an ideal of KO since, $\forall x \in KO$ and $a \in I$, the products $x * a \,\&\, a * x$ in KO are still elements of I.*
2) KO is not simple since, by 1) above, $I_1 = \{x_1\beta_1\}$ is an ideal.
3) KO is not semisimple since every ideal of KO is not simple.
4) KO is Jordan algebra since, $\forall x, y \in KO$,
i) $x * y = y * x$
and
ii) $x^2 * (y * x) = (x^2 * y) * x$ are satisfied.
5) KO is not a Malcev algebra since $x * y \neq -y * x, \forall x, y \in KO$.

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