International Journal of Mathematics and
Computer Science, **19**(2024), no. 3, 725–730

Deriving the Partial Fraction Series for the $\pi\cot(\pi x)$ Function by Applying Hypergeometric Functions Theory

Juan Carlos López Carreño[1], **Rosalba Mendoza Suárez**[1], **Jairo Alonso Mendoza Suárez**[2]

[1]Department of Mathematics
Faculty of Basic Sciences
University of Pamplona
Pamplona, Colombia

[2]Department of Physics-Geology
Faculty of Basic Sciences
University of Pamplona
Pamplona, Colombia

email: jairoam@unipamplona.edu.co

(Received December 11, 2023, Accepted January 13, 2024, Published February 12, 2024)

Abstract

In this article, we present a new proof of the partial fraction expansion of the cotangent function using the theory of hypergeometric functions.

1 Introduction

One of the most beautiful and useful series studied in classical analysis is the partial fraction decomposition of the cotangent function:

$$\pi\cot(\pi x) = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2x}{x^2 - k^2} \tag{1.1}$$

Key words and phrases: Partial fractions, hypergeometric series, cotangent function.
AMS (MOS) Subject Classifications: 11J83, 11K50, 11L03, 37E10, 60F05.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

The series for $\pi\cot(\pi x)$ was quite familiar to Euler who, in 1748 [1], discovered a proof by taking the logarithmic derivative of the formula for the infinite product of the sine function:

$$\sin(\pi x) = \pi x \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right).$$

(A similar test is shown in [2]).
In 1892, the German mathematician Friedrich Hermann Schottky [3] presented an elegant proof of the formula (1.1) using Eisenstein series. In the same line of reasoning, the German mathematician Gustav Herglotz observed that, in Schottky's proof, there is no need to apply the maximum principle at all times. Instead, he used what would later be known as the "Herglotz trick" which provides an astonishing simple demonstration of (1.1) [4]. The standard method of proving (1.1) can be found in complex variable texts as an application of Cauchy's residue theorem or as a consequence of the Mittag-Leffler representation theorem as the interested reader can see, for example, in [5], [6], [7], [8], [9]. Additionally, in [10], a proof is provided that employs some elements of harmonic analysis, such as the Fourier transform, the inversion theorem, and the Poisson summation formula. Finally, in [11], the author ingeniously demonstrates the result using only certain skillfully established trigonometric identities and some results from elementary calculus.

The objective of this article is to present a new proof of (1.1) using the theory of hypergeometric functions.

2 Hypergeometric Series

The series $\sum_{k=o}^{\infty} C_k$ is called a hypergeometric series if $\frac{C_{k+1}}{C_k}$ is the quotient of two polynomials in the variable k; that is,

$$\frac{C_{k+1}}{C_k} = \frac{(k+a_1)(k+a_2)\cdots(k+a_p)}{(k+b_1)(k+b_2)\cdots(k+b_q)}\frac{x}{(k+1)}. \tag{2.2}$$

Using (2.2), observe that

$$C_k = C_0\left(\frac{C_1}{C_0}\right)\left(\frac{C_2}{C_1}\right)\cdots\left(\frac{C_k}{C_{k-1}}\right). \tag{2.3}$$

The expression given in (2.3) can be transformed into

$$C_k = C_0\left(\frac{(a_1)(a_2)\cdots(a_p)x}{(b_1)(b_2)\cdots(b_q)1}\right)\left(\frac{(a_1+1)(a_2+1)\cdots(a_p+1)x}{(b_1+1)(b_2+1)\cdots(b_q+1)2}\right)$$
$$\cdots\left(\frac{(a_1+k-1)(a_2+k-1)\cdots(a_p+k-1)x}{(b_1+k-1)(b_2+k-1)\cdots(b_q+k-1)k},\right)$$

$$C_k = C_0\frac{(a_1)_k(a_2)_k\cdots(a_p)_k\,x^k}{(b_1)_k(b_2)_k\cdots(b_q)_k\,k!},$$

where $(\alpha)_k$ is the Pochhammer symbol defined as

$$(x)_n := x(x+1)(x+2)\cdots(x+n-1),\ \ n>0\,;(x)_0 := 1.$$

In this way, if $\sum_{k=o}^{\infty} C_k$, is a hypergeometric series, then we can write it as:

$$\sum_{k=o}^{\infty} C_k = C_0\sum_{k=o}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k\,x^k}{(b_1)_k(b_2)_k\cdots(b_q)_k\,k!}.$$

Using hypergeometric function notation, we have

$$\sum_{k=o}^{\infty} C_k = C_0\ {}_pF_q\left(\begin{array}{c} a_1,a_2,\cdots a_p \\ b_1,b_2,\cdots b_q \end{array}\middle| x\right).$$

In order to prove the identity (1.1), we need a summation formula for a hypergeometric function ${}_3F_2$, due to Lavoie [12]. This identity is expressed in terms of the Gamma function $\Gamma(x)$, defined by the limit

$$\Gamma(x) = \lim_{n\to\infty}\frac{n!\,n^{x-1}}{(x)_n}, \tag{2.4}$$

with $x\in\mathbb{C}$, $x\neq 0,-1,-2,\cdots$
Some important properties of the Gamma function are:

$$\Gamma(x+1) = x\Gamma(x),$$

which for the particular case of non-negative integer n becomes

$$\Gamma(n+1) = n!.$$

For all $x \in \mathbb{C}$, we have

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}.$$

The proof of the Addition formula

$$ {}_3F_2\left(\begin{matrix} a, b, c \\ 2+a-b, 2+a-c \end{matrix} \middle| 1 \right) = 2^{-2c+1} \frac{\Gamma(2+a-b)\,\Gamma(2+a-c)\,\Gamma(b-1)\,\Gamma(c-1)}{\Gamma(a-2c+2)\,\Gamma(a-b-c+2)\,\Gamma(b)\,\Gamma(c)} $$

$$ \times \left\{ \frac{\Gamma\left(\frac{a}{2}-c+\frac{3}{2}\right)\,\Gamma\left(\frac{a}{2}-b-c+2\right)}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}-b+1\right)} - \frac{\Gamma\left(\frac{a}{2}-c+1\right)\,\Gamma\left(\frac{a}{2}-b-c+\frac{5}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{a}{2}-b+\frac{3}{2}\right)} \right\}, \quad (2.5) $$

can be found in [12].

3 Demonstration of the formula (1.1)

Let

$$S = \sum_{k=1}^{\infty} \frac{1}{k^2 - x^2}. \quad (3.6)$$

To write this sum as a hypergeometric series, let us examine the quotient $\frac{C_{k+1}}{C_k}$. By calculation, we get

$$\frac{C_{k+1}}{C_k} = \frac{(k-x)(k+x)}{(k+1-x)(k+1+x)}, \quad (3.7)$$

taking into account the expression for C_k given in (2.3). The equality (3.7) allows us to write

$$C_k = C_1 \left(\frac{(-x+1)(x+1)}{(-x+2)(x+2)} \right) \left(\frac{(-x+2)(x+2)}{(-x+3)(x+3)} \right) \cdots \left(\frac{(-x+k-1)(x+k-1)}{(-x+k)(x+k)} \right),$$

$$C_k = C_1 \frac{(-x+1)_{k-1}(x+1)_{k-1}}{(-x+2)_{k-1}(x+2)_{k-1}}.$$

Then the series given in (3.6) can be written as

$$S = C_1 \sum_{k=1}^{\infty} \frac{(-x+1)_{k-1}(x+1)_{k-1}}{(-x+2)_{k-1}(x+2)_{k-1}},$$

using the change of variable $k-1=j$, and the identity $(1)_j = j\,!$. The above expression for S takes the form

$$S = C_1 \sum_{j=0}^{\infty} \frac{(-x+1)_j\,(x+1)_j\,(1)_j}{(-x+2)_j\,(x+2)_j\,j\,!},$$

that we recognize, according to the definition of a generalized hypergeometric function, as a ${}_3F_2$. More explicitly,

$$S = C_1 \;{}_3F_2\left(\begin{matrix} 1, -x+1, x+1 \\ x+2, -x+2 \end{matrix} \,\middle|\, 1 \right). \tag{3.8}$$

Using the addition formula (2.5) with the identification $a=1$, $b=-x+1$, $c=x+1$, the hypergeometric function that appears on the right side of (3.8) can be written as

$$S = C_1 \frac{2^{-2x-1}\,(1-x^2)\,\Gamma(x)\,\Gamma(-x)}{(-2x)\Gamma(-2x)} \times \left\{ \frac{\pi\,\Gamma(1+x)\,\Gamma(1-x) \;-\; \Gamma\left(\frac{1}{2}-x\right)\,\Gamma\left(\frac{1}{2}+x\right)}{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}+x\right)\,\Gamma\left(1+x\right)} \right\}.$$

Taking into account the properties of the gamma function, we obtain

$$\begin{aligned} S &= C_1 \frac{2^{-2x-1}(1-x^2)\left(\frac{-\pi}{x\sin(\pi x)}\right)}{(-2x)\Gamma(-2x)\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(1+x\right)\Gamma\left(\frac{1}{2}\right)} \left\{ \frac{\pi^2 x}{\sin(\pi x)} \;-\; \frac{\pi}{\cos(\pi x)} \right\} \\ &= C_1 \frac{(1-x^2)\cos(\pi x)}{(-2x)^2} \left\{ \frac{\pi x}{\sin(\pi x)} \;-\; \frac{1}{\cos(\pi x)} \right\}, \end{aligned}$$

and so

$$S = C_1 \frac{(1-x^2)}{(-2x)^2} \left\{ \pi x \cot(\pi x) \;-1 \right\}. \tag{3.9}$$

From (3.7), (3.8), and (3.9), and noting that $C_1 = \frac{1}{1-x^2}$, we conclude that

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k^2-x^2} &= \frac{1}{2x^2}\left[1-\pi x\cot(\pi x)\right], \\ \sum_{k=1}^{\infty} \frac{2x}{k^2-x^2} &= \frac{1}{x} - \pi\cot(\pi x), \\ \pi\cot(\pi x) &= \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2x}{x^2-k^2} \qquad \square \end{aligned}$$

4 Conclusion

Using the theory of hypergeometric functions, we gave a proof of Euler's well-known formula for the expression of the cotangent function as a series of partial fractions.

References

[1] Leonard Euler, *Introductio in analysin infinitorum,* **1,** (1748).

[2] George E. Andrews, Richard Askey, Ranjan Roy, *Special Functions*, Encyclopedia of Mathematics and its Applications, **71,** Cambridge University Press, 2000.

[3] F.H. Schottky, *Über das Additionstheorem der Cotangente,* Jour. für Reine u. Angew. Math., **110,** (1892), 324–337.

[4] Martin Aigner, Günter M. Ziegler, *Proofs from THE BOOK,* Springer, (2010).

[5] Lars V. Ahlfors, *Complex Analysis,* McGraw-Hill, New York,(1979), 187–189.

[6] James W. Brown, Ruel V. Churchill, *Complex variables and Applications,* McGraw-Hill, New York, (1984), 277–280.

[7] John B. Conway, *Functions of one Complex Variable,* Springer, New York, (1978), 250–260.

[8] E. C. Titchmarsh, *Theory of Functions,* Oxford University Press, 1939, 112-113.

[9] B. P. Palka, *An introduction to Complex function Theory,* Springer-Verlag, New York, 1991, 484-486.

[10] Walter Rudin, *Real and Complex Analysis,* McGraw-Hill, New York, 1987, 316.

[11] Paul Loya, *Amazing and Aesthetic Aspects of Analysis,* Springer, New York, 2017, 556–559.

[12] J. Lavoie, F. Grondin, A. Rathie, K. Arora, *Generalizations of Dixon's theorem on the sum of a* ${}_3F_2$, Math. Comp., **62,** (1994), 267–276.