International Journal of Mathematics and
Computer Science, **19**(2024), no. 3, 605–611

A higher-order family of simultaneous iterative methods with Neta's correction for polynomial complex zeros

Roselaine Neves Machado[1], **Luiz Guerreiro Lopes**[2]

[1]Federal Institute of Rio Grande do Sul
Bento Gonçalves Campus
95700-206 Bento Gonçalves, RS, Brazil

[2]Faculty of Exact Sciences and Engineering
University of Madeira, Penteada Campus
9020-105 Funchal, Madeira Is., Portugal

email: roselaine.machado@bento.ifrs.edu.br, lopes@uma.pt

(Received November 6, 2023, Accepted November 29, 2023,
Published February 12, 2024)

Abstract

In this paper, a new family of iterative methods for the simultaneous approximation of simple complex polynomial zeros is presented. The proposed family of simultaneous methods is constructed on the basis of the well-known third order Ehrlich iteration, combined with an iterative correction from the sixth order Neta's method for nonlinear equations. It is proved that the use of this iterative correction allows to increase the convergence order of the basic method from three to eight. Numerical examples are given to illustrate the convergence and effectiveness of the proposed combined method.

1 Introduction

Given the significance of finding polynomial zeros in pure and applied sciences and engineering, many numerical methods have been developed for this purpose, and novel and efficient iterative methods are also of interest.

Key words and phrases: Polynomial zeros, simultaneous iterative methods, Ehrlich–Aberth method, Neta's sixth order method.
AMS (MOS) Subject Classifications: 30C10, 30C15, 65H04.
The corresponding author is Luiz Guerreiro Lopes.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

These methods may estimate polynomial zeros sequentially or simultaneously. The sequential determination of all zeros of a polynomial requires repeated deflations, which could yield erroneous results caused by rounding errors in finite precision floating-point arithmetic. In turn, the simultaneous methods are intrinsically parallel and prevent deflation, but they require good starting guesses for all zeros to converge.

This paper presents a new family of iterative methods for simultaneous approximation of simple complex polynomial zeros, based on the well-known third-order Ehrlich iteration [3], combined with an iterative correction from the sixth-order Neta's method for nonlinear equations [5], allowing to increase the convergence order of the basic simultaneous method from three to eight.

2 A new family of Ehrlich-type methods with Neta's correction

2.1 Preliminary result

Let P be a monic polynomial of degree n, with real or complex simple zeros $\zeta_1, \ldots, \zeta_n$, i.e., $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 = \prod_{j=1}^{n}(z - \zeta_j),\ a_i \in \mathbb{C}$.

By taking the logarithmic derivative of P with respect to z and the correction term $N(z) = P(z)/P'(z)$ from the 2nd order Newton's method $\hat{z} = z - N(z)$, where $\hat{z}$ is a new estimate for ζ, we obtain $N(z) = (\sum_{j=1}^{n} \frac{1}{z-\zeta_j})^{-1}$, and then $N(z) = (\frac{1}{z-\zeta_i} + \sum_{j=1,\, j\neq i}^{n} \frac{1}{z-\zeta_j})^{-1}$, which leads to the fixed-point relation $\zeta_i = z - (\frac{1}{N(z)} - \sum_{j=1,\, j\neq i}^{n} \frac{1}{z-\zeta_j})^{-1}$, $i = 1, \ldots, n$.

By setting $\zeta_i \simeq \hat{z}_i$ and $z = z_i$, where $\hat{z}_i$ are the updated values of the approximations z_i to the zeros ζ_i $(i = 1, \ldots, n)$, and replacing ζ_j by their approximations z_j $(j \neq i)$ in the fixed-point relation above, we get the Ehrlich iteration method [3] for simple polynomial zeros, given by:

$$\hat{z}_i = z_i - \frac{1}{\dfrac{1}{N(z_i)} - \displaystyle\sum_{j=1,\, j\neq i}^{n} \frac{1}{z_i - z_j}} \quad (i = 1, \ldots, n). \tag{2.1}$$

This third-order iteration (see, e.g., [2]), also derived by Aberth [1] and some other authors, is also referred to as the Ehrlich–Aberth method.

2.2 Ehrlich-type methods with Neta's correction

Examination of the preceding fixed-point relation shows that better estimates z_j will result in more accurate approximations $\hat{z}_i$ for the zeros ζ_i. In order to achieve this, we propose using the approximations $\zeta_j = z_j - K_N(z_j)$ $(j \neq i)$ on the right hand side of that identity, where $K_N(z_j)$ is a correction term from Neta's sixth-order family of methods for nonlinear equations [5].

The proposed family of Ehrlich-type simultaneous iterative methods, constructed on the basis of the third-order Ehrlich method (2.1), is defined by:

$$\hat{z}_i = z_i - \frac{1}{\dfrac{1}{N(z_i)} - \displaystyle\sum_{j=1,\, j\neq i}^{n} \frac{1}{z_i - z_j + K_N(z_j)}} \quad (i = 1, \ldots, n). \tag{2.2}$$

The correction term $K_N(z_j)$ in (2.2) is given by:

$$K_N(z_j) = \frac{P(y_j)}{P'(z_j)} \frac{P(z_j) - P(x_j)}{P(z_j) - 3\,P(x_j)}, \tag{2.3}$$

where

$$x_j = z_j - \frac{P(z_j)}{P'(z_j)} \tag{2.4}$$

and

$$y_j = x_j - \frac{P(x_j)}{P'(z_j)} \frac{P(z_j) + \alpha\,P(x_j)}{P(z_j) + (\alpha - 2)\,P(x_j)}, \tag{2.5}$$

with α as a parameter.

As proved below, the convergence order of (2.2) is eight.

2.3 Convergence of the family of combined methods

Theorem 2.1. *Let $z_1^{(0)}, \ldots, z_n^{(0)}$ be initial approximations sufficiently close to the zeros $\zeta_1, \ldots, \zeta_n$ of the polynomial P. Then, the order of convergence of the one-parameter family of iterative methods defined in* (2.2) *is eight.*

Proof. Let $\epsilon_i = z_i - \zeta_i$ and $\hat{\epsilon}_i = \hat{z}_i - \zeta_i$ be the numerical approximation errors and, for convenience, be the following abbreviations:

$$\phi_{i,j} = z_i - z_j + K_N(z_j), \tag{2.6}$$

$$\psi_i = \sum_{j=1,\, j\neq i}^{n} \frac{z_j - \zeta_j - K_N(z_j)}{(z_i - \zeta_j)\,\phi_{i,j}}. \tag{2.7}$$

Using the expression for $N(z)$ obtained in Section 2.1 into (2.2), we get:

$$\hat{z}_i = z_i - \frac{1}{\displaystyle\sum_{j=1}^{n} \frac{1}{z_i - \zeta_j} - \sum_{j=1,\, j\neq i}^{n} \frac{1}{z_i - z_j + K_N(z_j)}} \quad (i = 1, \ldots, n). \tag{2.8}$$

When $i \neq j$, the first sum can be divided into two parts, which gives:

$$\hat{z}_i = z_i - \frac{1}{\displaystyle\frac{1}{z_i - \zeta_i} + \sum_{j=1,\, j\neq i}^{n} \frac{1}{z_i - \zeta_j} - \sum_{j=1,\, j\neq i}^{n} \frac{1}{z_i - z_j + K_N(z_j)}}. \tag{2.9}$$

Given (2.6) and the previously defined approximation errors, we obtain:

$$\hat{\epsilon}_i = \epsilon_i - \frac{\epsilon_i}{1 + \epsilon_i \left(\displaystyle\sum_{j=1,\, j\neq i}^{n} \frac{1}{z_i - \zeta_j} - \sum_{j=1,\, j\neq i}^{n} \frac{1}{\phi_{i,j}} \right)}. \tag{2.10}$$

Combining the two sums and taking (2.6) again into account, we have:

$$\hat{\epsilon}_i = \epsilon_i - \frac{\epsilon_i}{1 + \epsilon_i \left(\displaystyle\sum_{j=1,\, j\neq i}^{n} \frac{K_N(z_j) - z_j + \zeta_i}{(z_i - \zeta_j)\, \phi_{i,j}} \right)}. \tag{2.11}$$

Now, taking (2.7) and reducing to the common denominator results in:

$$\hat{\epsilon}_i = \frac{-\epsilon_i^2\, \psi_i}{1 - \epsilon_i\, \psi_i}. \tag{2.12}$$

By the theorem's assumption, the initial approximations are sufficiently close to the zeros, and so the errors ϵ_i and $\hat{\epsilon}_i$ are small enough in modulus. Thus, we can assume that $\epsilon_i = \mathcal{O}_m(\epsilon_j)$ and $\hat{\epsilon}_i = \mathcal{O}_m(\hat{\epsilon}_j)$, i.e., $|\epsilon_i| = \mathcal{O}(|\epsilon_j|)$ and $|\hat{\epsilon}_i| = \mathcal{O}(|\hat{\epsilon}_j|)$ for $i, j \in \{1, \ldots, n\}$.

Analysis of (2.6) and (2.7) shows that the denominator in (2.7) is limited and tends to $(\zeta_i - \zeta_j)^2$ for $i \neq j$ for approximations close enough to the zeros. On the other hand, the order of convergence of Neta's method is 6, that is, $\hat{z}_i - \zeta = \mathcal{O}_m((z_i - \zeta)^6)$.

Taking these last two results into account, we find that $\psi_i = \mathcal{O}_m(\epsilon^6)$. By applying this result to (2.12), we can finally conclude that $\hat{\epsilon} = \mathcal{O}_m(\epsilon^8)$, which proves that the convergence order of the one-parameter family of simultaneous methods with Neta's correction (2.2) is equal to eight. $\square$

3 Numerical examples

Two numerical examples are given below to illustrate the convergence and effectiveness of the proposed eighth-order Ehrlich-like family of combined methods for the simultaneous approximation of simple zeros of polynomials.

As in [5], the value $\alpha = -\frac{1}{2}$ was here adopted for the parameter in (2.5). The initial guesses were obtained using Aberth's initialization scheme [1], adopting an inclusion radius r for the zeros given by Guggenheimer's upper bound [4]. The results were computed using double-precision floating-point arithmetic, with a numerical tolerance of 1×10^{-12} and a maximum number of 50 iterations.

Example 1. The first example considered here for illustration purposes, taken from [6], is a complex polynomial of degree 5 with zeros $-1, 1 \pm 2i, 3, 5i$: $P_1(z) = z^5 - (4+5i)z^4 + (6+20i)z^3 - (4+30i)z^2 - (15-20i)z + 75i$.

Figure 1 (left) presents the approximation trajectories for P_1 generated by (2.2) with $\alpha = -1/2$ and $r \approx 12.806248474865697$. The requested accuracy was achieved in 8 iterations, while Ehrlich's method (2.1), which served as the basis for the proposed simultaneous method, required 12 iterations.

Example 2. The second illustrative example, extracted from [7], is that of a real polynomial of degree 15 with only a real zero ($\approx -1.146854042199507$): $P_2(z) = z^{15} + z^{14} + 1$.

The approximation trajectories for the polynomial P_2 generated by (2.2) with $\alpha = -1/2$ and $r = 2$ are shown in Figure 1 (right).

Convergence in this case was achieved in 9 iterations, while Ehrlich's method needed 14 iterations to reach the required accuracy.

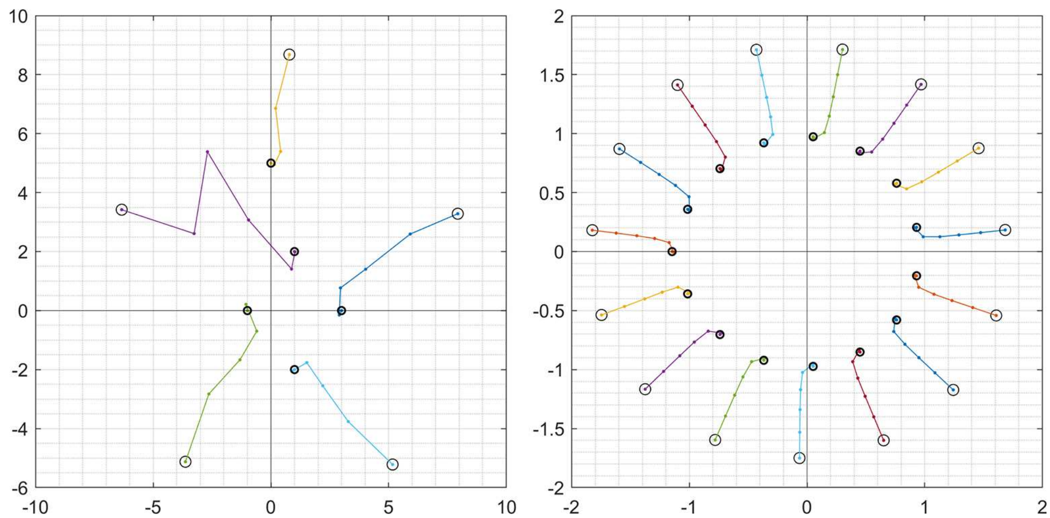

Figure 1: Approximation trajectories for Examples 1 (left) and 2 (right).

4 Conclusion

The proposed one-parameter family of simultaneous iterative methods for polynomial complex zeros, which was built from the third-order Ehrlich iteration and a correction term from Neta's sixth-order family of iterative methods for nonlinear equations, allowed the basic simultaneous method's convergence order to be increased from three to eight.

The numerical examples presented illustrate the rapid convergence and effectiveness of the proposed family of combined iterative methods, thus corroborating the theoretical analysis of their order of convergence.

Acknowledgment. This work was supported by the Federal Institute of Rio Grande do Sul (IFRS), Brazil. The content of this paper was partially presented at the International Conference on Mathematical Studies and Applications held in Karaman, Turkey, on October 4–6, 2018.

References

[1] O. Aberth, Iteration methods for finding all zeros of a polynomial simultaneously, Math. Comput., **27**, no. 25, (1973), 339–344.

[2] G. Alefeld, J. Herzberger, On the convergence speed of some algorithms for the simultaneous approximation of polynomial roots, SIAM J. Numer. Anal., **11**, no. 2, (1974), 237–243.

[3] L. W. Ehrlich, A modified Newton method for polynomials, Comm. ACM, **10**, no. 2, (1967), 107–108.

[4] H. Guggenheimer, Bounds for roots of algebraic equations, Arch. Math., **31**, no. 1, (1978), 568–569.

[5] B. Neta, A sixth-order family of methods for nonlinear equations, Int. J. Comput. Math., Section B, **7**, no. 2, (1979), 157–161.

[6] M. S. Petković, L. V. Stefanović, On the simultaneous method of the second order for finding polynomial complex zeros in circular arithmetic, Freiburger Intervall-Berichte, **85,** no. 3, (1985), 63–95.

[7] P. D. Proinov, M. T. Vasileva, On the convergence of high-order Ehrlich-type iterative methods for approximating all zeros of a polynomial simultaneously, J. Inequal. Appl., (2015), Article no. 336.