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The Action of the Chevalley Group E_6 on the Singular Subspaces V_2 of a 27-Dimensional Module Over a Field of Characteristic 2

Abdulkareem Alhuraiji, Mashhour Al-Ali Bani-Ata

Department of Mathematics
Faculty of Basic Education
Public Authority for Applied Education and Training
Ardiyah, Kuwait

email: abdulkaremalhuraiji@gmail.com, mashhour_ibrahim@yahoo.com

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Abstract

The purpose of this paper is to use the notion of the generalized quadrangle $(\Omega, \mathcal{L})$ of type $O_6^-(2)$ to show that $N_E(A(B))$ induces $SL(A(B))$ on $A(B)$, where B is a coclique of $\Omega, A(B) = \langle e_x \mid x \in B \rangle$ is the space spanned by $e_x, x \in B$ and $N_E(A(B))$ is the stabilize of $A(B)$ in the Chevalley group $E = E_6(K)$ of type E_6 over a field K of characteristic 2, and to show that E acts transitively on the family V_2 of all 2-dimensional singular subspaces of the 27-dimensional module A over K, we also compute $|V_2|$, the order of V$_2$. Next, we compute the action of the subgroup $H = \langle U_\Delta, U_{\Delta^\sigma} \rangle$ of E on the 27-dimensional module A over K, where $U_\Delta, U_{\Delta^\sigma}$ are two root subgroups of E and we prove that H acts completely reducibly on A with $H \cong SL_2(K)$. It is important to mention that the motivation behind this work is the challenging nature of the investigation of Chevalley groups, in particular E_6, over fields of characteristic 2, and also that this work could lead to more understanding of this type of group.

Key words and phrases: Generalized quadrangle, coclique, root subgroups, root base, root element.
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The corresponding author is Mashhour Al-Ali Bani-Ata.
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1 Introduction

Algebraic structures, particularly the interplay between generalized quadrangles and Chevalley groups, have been the subject of profound investigation in the field of mathematics. This paper delves into the intricacies of a specific case, namely, the generalized quadrangle $(\Omega, \mathcal{L})$ of type $O_6^-(2)$, and its profound connections with the Chevalley group $E_6(K)$ over a field K with characteristic 2. The study initiates by considering the stabilizer $N_E(A(B))$ within the context of $E_6(K)$. Our primary objective is to establish that this stabilizer induces $SL(A(B))$ on the vector space $A(B)$, where B denotes a coclique of the generalized quadrangle Γ. The vector space $A(B)$, spanned by vectors e_x for $x \in B$, emerges as a critical focus, revealing essential insights into the algebraic structure under consideration. Moving beyond the stabilizer analysis, our investigation extends to the transitive action of the Chevalley group E on the family V_2. This family encompasses all 2-dimensional singular subspaces of the 27-dimensional module A over the field K. The determination of the order $|V_2|$ offers quantitative measures that illuminate the geometric properties embedded within the algebraic framework. A significant portion of our study is dedicated to exploring the action of the subgroup $H = \langle U_\Delta, U_\Delta^\sigma \rangle$ of $E_6(K)$ on the 27-dimensional module A. Here, U_Δ and $U_{\Delta\sigma}$ represent distinct root subgroups of $E_6(K)$. Our rigorous analysis establishes that this subgroup H acts completely reducibly on A, ultimately revealing an isomorphism with $SL_2(K)$. This paper not only contributes to the broader understanding of the mathematical relationships intrinsic to the generalized quadrangle of type $O_6^-(2)$ and its interaction with the Chevalley group $E_6(K)$ but also provides a foundation for further exploration into the profound implications of these algebraic structures. our main upshot is to leverage the notion of the generalized quadrangle $(\Omega, \mathcal{L})$ of type $O_6^-(2)$ to establish that the stabilizer $N_E(A(B))$ induces $SL(A(B))$ on the vector space $A(B)$, where B represents a coclique of Ω, and $A(B) = \langle e_x | x \in B \rangle$ is the space spanned by vectors e_x for $x \in B$. The study is conducted within the context of the Chevalley group $E = E_6(K)$ of type E_6 over a field K with characteristic 2. Additionally, the paper aims to demonstrate the transitive action of E on the family V_2, encompassing all 2- dimensional singular subspaces of the 27-dimensional module A over K. The order of V_2, denoted as $|V_2|$, is computed. Furthermore, the paper explores the action of the subgroup $H = \langle U_\Delta, U_\Delta^\sigma \rangle$ of E on the 27-dimensional module A over K, where U_Δ and $U_{\Delta\sigma}$ are two root subgroups of E. It is proven that H acts completely reducibly on A, and H is isomorphic to $SL_2(K)$. For more information about E_6, one may refer to [5,6,7,8,9,10,11,12,13].

2 Notation and Earlier Results.

Let V be a 6-dimensional vector space over $\mathbb{F}_2$ and Q be a non-degenerate quadratic form on V of minimal Witt-index. Define a bilinear form on V associated with Q, by $\langle v|w) = Q(v+w) + Q(v) + Q(w)$. Let Ω be the set of non-zero singular elements of V and $\mathcal{L}$ be the set of the two-dimensional singular subspaces of V. The elements of Ω are called points and the elements of $\mathcal{L}$ are called lines. There are 27 points, 45 lines and 36 vectors $s \in V$ with $Q(s) = 1$, called exterior vectors. The pair $(\Omega, \mathcal{L})$ is called a generalized quadrangle with 27 points and 45 lines, each line contains 3 points and each point lies on exactly 5 lines. The group $W = \{g \in GL(V) \mid Q(x^g) = Q(x), \forall x \in V\}$ is the Weyl group generated by the 36 -reflections σ_s, s is an exterior point, defined by:

$$v^{\sigma_s} = v + (v \mid s)s \text{ for } v \in V.$$

Definition 2.1. A proper subset Δ of Ω is a root base, if Δ is a $\mathbb{F}_2$- base of V and Δ is a coclique, $i \cdot e \quad (x \mid y) = 1$ for distinct elements $x, y \in \Delta$. There are exactly 72 root bases. Let A be a 27 -dimensional vector space over the field K of characteristic 2, with bases $e_x, x \in \Omega$. Let $\langle \mid \rangle$ be a bilinear form on A such that $\{e_x \mid x \in \Omega\}$ is an orthogonal base i.e

$$\langle e_x \mid e_y \rangle = \begin{cases} 1, & x = y \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2.1. *The vector space A can be turned into a non-associative, distributive algebra over K by defining the multiplication*

$$e_x e_y = \begin{cases} e_z, & \{x, y, z\} \in \mathcal{L} \\ 0, & \textit{otherwise} \end{cases}$$

Definition 2.2. For a root base $\Delta \subseteq \Omega$ and $k \in K$, define the root elements $r_\Delta(k) \in GL(A)$ by

$$e_x^{r_\Delta(k)} = \begin{cases} e_x + k e_{x^{\sigma_\Delta}} & , \quad x \in \Delta \\ e_x & , \quad \textit{otherwise} \end{cases}$$

where $\sigma_\Delta := \sigma_{s_\Delta}$ and $s_\Delta = \sum_{x \in \Delta} x$, As $Q(s_\Delta) = 1$, then s_Δ is an exterior vector.

The group $E = \langle U_\Delta(k) | \, \Delta \text{ root base}, \; k \in K\rangle$ is the Chevalley group of type E_6 and $U_\Delta(k)$is the root subgroup $U_\Delta(k) = \langle r_\Delta(k) \mid k \in K\rangle \cong K$
For the above notations see [1, 2, 3].

Definition 2.3. Define the quadratic map $Q_p, p \in \Omega$, on A by

$$Q_p(a) = \sum_{\{p,x,y\} \in \mathcal{L}} a_x a_y$$

where $a = \sum_{x \in \Omega} a_x e_x$ and the quadratic map $\hat{Q} : A \to A$ by $\hat{Q}(a) = \sum_{p \in \Omega} Q_p(a) e_p$

Proposition 2.1. *[8]. Let $a, b \in A$, then $\hat{Q}(a+b) = \hat{Q}(a) + \hat{Q}(b) + ab$.*
Note : *A subspace $U < V$ is singular with respect to $\hat{Q}$ if $\hat{Q}(U) = 0$.*

3 Results

Proposition 3.1. *Let $B \subseteq \Omega$ be a coclique and let $A(B) = \langle e_x \mid x \in B \rangle$. Then $A(B)$ is singular and $N_E(A(B))$ induces $SL(A(B))$ on $A(B)$.*

Proof. From the definition of $Q_p(a)$, it follows that $A(B)$ is singular, as $e_x e_y = 0$ for all $x, y \in$ B.

For $x, y \in B, x \neq y$, consider the root base $\triangle$ with $s_\Delta = x + y$. If $z \in B \backslash \{x, y\}$, then $(z \mid x + y) = 0$ which implies

$$e_z^{r_\Delta(k)} = e_z \text{ for } r_\Delta(k) \in E \text{ and } k \in K$$

and

$$e_x^{r_\Delta(k)} = \begin{cases} e_x + k e_{x+s_\Delta} = e_x + k e_y, & x \in \Delta \\ e_x & \text{, otherwise} \end{cases}$$

Let $\Delta^* = \Delta + s_\Delta$ be the root base corresponding to Δ then

$$e_x^{r^*_\Delta(k)} = \begin{cases} e_x + k e_{y+s_\Delta} = e_y + k e_x, & y \in \Delta^* \\ e_y & \text{, otherwise} \end{cases}$$

Hence the elements $r_\Delta(k)$ with $s_\Delta = x + y, x \neq y$, induce all elementary transvections on $A(B)$, and hence generate $SL(A(B))$ on $A(B)$. $\square$

Remark 3.1. *Let $V_i = \{U \leqslant A \mid \dim U = i$ and $\hat{Q}(U) = 0\}$, then $V_i \neq \varnothing$ if $i \leqslant 6$. For the proof see [2].*

Proposition 3.2. *Let $V_i = \{U \leq A \mid \dim U = i$ and $\hat{Q}(U) = 0\}$, and let $\hat{U} \in V_i, \hat{V} \in V_{i+1}$ and $\hat{U} < \hat{V}$, then $\hat{V} = \hat{U} + ka$ for some $a \in A \backslash \hat{V}$ with $\hat{Q}(a) = 0$ and $\hat{U} \cdot a = 0$*

Proof. Pick out $a \in \hat{V} \backslash \hat{U}$, such that $\hat{V} = \hat{U} + ka$. As $\hat{V} \in V_{i+1}$, this implies $\hat{Q}(a) = 0$ and $\hat{Q}(u) = 0$ for all $u \in \hat{U}$. As $u + a \in \hat{V} \in V_{i+1}$, it follows that $\hat{Q}(u + a) = \hat{Q}(u) + \hat{Q}(a) + ua = ua$ by Proposition 1.1, hence the claim.

$\square$

Theorem 3.1. *The Chevalley group E is transitive on V_2 and on the set $\{(U_1, U_2)$ such that $U_1 < U_2, U_i < V_i,\ i = 1, 2\}$.*

Proof. Let $U \in V_2$. Pick out $0 \neq a \in U \in V_2$, then $\hat{Q}(a) = 0$. Hence $\langle a \rangle \in V_1$. As E is transitive on V_1 [4], it follows that there exists $g \in E$ with $a^g = e_p$ for $p \in \Omega$. Hence, $e_p \in U^g$. With out loss of generality, one may assume that $e_p \in U$. Hence $U = \langle e_p, a\rangle$ and $e_p a = 0$ by Proposition 2.2 and Proposition 1.1. This means that e_p and a are independent as $\hat{Q}(a) = \hat{Q}\left(e_p\right) = 0$. Thus, $a \in \{z \in A \mid e_p z = 0\} = \langle e_p, e_x \mid x \in \Omega, (p \mid x) = 1\rangle := A_1(p)$. In [4] , it was proven that $N_E(\langle e_p \rangle)$ is transitive on $\left\{0 \neq a \in A_1(p) \mid \hat{Q}(a) = 0\right\}$, so $N_E\left(\langle e_p \rangle\right)$ is transitive on $\{U \mid U \in V_2,\ e_p \in U\}$ and E is transitive on V_2 and on all pairs $\{U_1, U_2) \mid U_1 < U_2,\ U_i \in V_i, i = 1, 2\}$; Hence the claim. $\square$

Proposition 3.3. *If $K = \mathbb{F}_q$, then $|V_2| = \frac{1}{q-1} \cdot \frac{\left(q^{12}-1\right)\left(q^{9}-1\right)\left(q^{8}-1\right)\left(q^{6}-1\right)}{(q^4-1)(q^3-1)(q^2-1)}$*

Proof. As E is transitive on pairs $\left(U_2, U_3\right), U_2 < U_3 \in V_3$, one obtains $|V_1| \cdot r_{12} = |V_2| \cdot (q+1)$ by counting on two ways the number of subspaces in V_2 containing the the 1-dimensional subspaces $\langle a \rangle \in V_1$ with $\hat{Q}(a) = 0$, where r_{12} is the number of $U_2 \in V_2$ containing a given $0 \neq a \in A$ with $\hat{Q}(a) = 0$. Taking $a = e_p$, $p \in \Omega$, one has
$r_{12} = \frac{1}{q-1} \left| \left\{0 \neq a_1 \in A_1(p) \mid \hat{Q}(a) = 0\right\} \right|$
$= \frac{\left(q^8-1\right)\left(q^3+1\right)}{q-1}$, see [9] .
Hence $|V_2| = \frac{|V_1|\left(q^8-1\right)\left(q^3+1\right)}{(q^2-1)(q^4-1)(q-1)}$, where $|V_1| = \frac{|\left(q^{12}-1\right)(q^9-1)|}{(q^4-1)(q-1)}$[4].

$$= \frac{\left(q^{12}-1\right)\left(q^{9}-1\right)\left(q^{8}-1\right)\left(q^{3}+1\right)}{\left(q^{4}-1\right)\left(q^{2}-1\right)\left(q-1\right)}$$
$$= \frac{\left(q^{12}-1\right)\left(q^{9}-1\right)\left(q^{8}-1\right)\left(q^{6}-1\right)}{\left(q^{4}-1\right)\left(q^{3}-1\right)\left(q^{2}-1\right)\left(q-1\right)}.$$

Hence the claim. $\square$

4 The Action of $H = \langle U_\Delta, U_{\Delta^\sigma} \rangle := \left\langle U_\Delta, U_{\Delta + s_\Delta} \right\rangle$ on A.

Theorem 4.1. *Let Δ be a root base and let σ be the reflection correspoding to Δ. Then $\langle U_\Delta, U_{\Delta^\sigma} \rangle \cong SL_2(K)$.*

Proof. If $x \in \Delta$, then $e_x^{r_\Delta(k)} = e_x + ke_{x^\sigma}$ and $e_x^{r_\Delta(k)} = e_x$ as $x \notin \Delta^\sigma$. If $x \in \Delta^\sigma = \Delta + s_\Delta$, then $e_x^{r_\Delta(k)} = e_x$ and $e_x^{r_\Delta(k)} = e_x + ke_{x^\sigma}$
Hence $\langle U_\Delta, U_{\Delta^\sigma} \rangle$ leaves the space $\langle e_x, e_{x^\sigma} \rangle$ invariant for all $x \in \Delta$. In a matrix form, the above action can be represented as follows:

$$r_\Delta(k): \begin{array}{c} e_x \\ e_{x^\sigma} \end{array} \begin{bmatrix} 1 & & k \\ 0 & & 1 \end{bmatrix}, \quad r_{\Delta^\sigma}: \begin{array}{c} e_x \\ e_{x^\sigma} \end{array} \begin{bmatrix} 1 & & 0 \\ k & & 1 \end{bmatrix}$$

If $y \notin \Delta \cup \Delta^\sigma$, then $e_y^{r_\Delta(k)} = e_y = e_y^{r_{\Delta^\sigma}(k)}$. So $\langle U_\Delta, U_{\Delta^\sigma} \rangle$ induces $SL_2(K)$ on $\langle e_x, e_{x^\sigma} \rangle$.
Hence, the action of $\langle U_\Delta, U_{\Delta^\sigma} \rangle$ on A is a direct sum of 6-dimensional standard submodules and a trivial submodule of dimension 15, as $\Omega = \Delta \cup \Delta^\sigma \cup \Delta_0$, where $\Delta_0 = \{x \in \Omega \mid (x \mid s_\Delta) = 0\}, |\Delta_0| = 15$. To show that it acts completely reducibly on A, we prove the following proposition. □

Proposition 4.1. *Let Δ, Γ be two root bases with reflections $\sigma_\Delta, \sigma_\Gamma$ respectively. Let $Y = \langle \sigma_{\Delta,}, \sigma_\Gamma \rangle$. If R is an orbit of Y on Ω, then A decomposes as Y-modules ie $A = \bigoplus\limits_R A(R)$*

Proof. As Y acts on Ω, let $x \in R$, then $e_x^{r_\Delta(k)} = e_x + ke_{x^\sigma} \in A(R)$ as $x, x^{\sigma_\Delta} \in R$.
Hence the claim. □

Corollary 4.1. *For $H = \langle U_\Delta, U_{\Delta^\sigma} \rangle, \sigma_\Delta = \Delta^\sigma$. This implies $Y \cong C_2$ and Y has 6 orbits of length 2 and 15 fixed points. Hence it acts faithfully on each factor of $\dim 2$ i.e $A \cong 2^6 \oplus 1^{15}$. This completes the proof of Theorem 3.1. Compare with Theorem 2.2[9].*

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