International Journal of Mathematics and
Computer Science, **19**(2024), no. 3, 697–703

On a Radical of Nearrings Which is Hereditary

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(Received December 1, 2023, Accepted January 8, 2024,
Published February 12, 2024)

Abstract

We introduce and study properties of a radical in near-rings which is a generalization of the Jacobson radical of rings. Moreover, we proved that this radical is hereditary. Furthermore, we compare this radical with the existing Jacobson type radicals of near-rings.

Key words and phrases: near-ring, S-group, S-group of type-2^*, J_{2^*} radical, KA-radical.
AMS (MOS) Subject Classifications: 16Y30.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

1 Introduction

Nearrings are right nearrings. Jacobson radicals of nearrings J_2 and J_3 are Kurosh-Amitsur radicals in zerosymmmetric nearrings and are also ideal-hereditary in the same class ([3], [2]). Moreover, J_2 fails to be Kurosh-Amitsur in nearrings. It is not known whether J_3 is Kurosh-Amotsur in nearrings ([6]).
In this paper, we introduce a (left) Jacobson radical J_{2^*} in nearrings and proved that J_{2^*} is ideal-hereditary in zerosymmetric nearrings. Also, for a nearring S, $J_2(S) \subseteq J_{2^*}(S) \subseteq J_3(S)$. It is known that for a nearring S, $J_2(S) \subseteq J_{5/2}(S) \subseteq J_3(S)$ ([5]). It is established that the radicals $J_{5/2}, J_{2^*}$ are independent. Even though an S-group of type 2^* has the basic characteristics of an S-group of type 3, some natural nearrings are presented which could be retained as J_{2^*}-semisimple nearrings even though they are J_3-radical nearrings.
In this paper, S stands for a right nearring and S_0 is the zerosymmetric part of S. A group $(\Gamma, +)$ is a (left) *S-group* if there is mapping $(s, \gamma) \to s\gamma$ of $S \times \Gamma$ into Γ such that:

(i) $(s+t)\gamma = s\gamma + t\gamma$;

(ii) $(st)\gamma = s(t\gamma)$ for all $s, t \in S, \gamma \in \Gamma$.

Consider an S-group Γ. A normal subgroup Δ of $(\Gamma, +)$ is an *ideal* of the S-group Γ if $s(\gamma + \delta) - s\gamma \in \Delta$ for all $s \in S, \gamma \in \Gamma, \delta \in \Delta$. Also a subgroup Δ of $(\Gamma, +)$ is an *S-subgroup* of the S-group Γ if $s\delta \in \Delta$ for all $s \in S, \delta \in \Delta$. $\gamma \in \Gamma$ is a *generator* of the S-group Γ if $S\gamma = \Gamma$. S-group Γ is *monogenic* if it has a generator. An S-group Γ is *S-simple* if it has no S-subgroups except $S0$ and Γ.
A monogenic S-group Γ with $\Gamma \neq \{0\}$ is *type* 2 if Γ is S_0-simple.
Γ is of *type* 5/2 if it of type 2 and $S\gamma = \Gamma$ for all $0 \neq \gamma \in \Gamma$ ([5]).
An S-group Γ is of *type*3 if it of type 2 and $\gamma_1, \gamma_2 \in \Gamma$ and $s\gamma_1 = s\gamma_2$ for all $s \in S$ implies $\gamma_1 = \gamma_2$.
Note that an S-group of type3 is of type-5/2 and an S-group of type-5/2 is of type 2.
A mapping ϱ on nearrings such that $\varrho(S)$ is an ideal of S for all nearrings S is an *ideal-mapping.*
A *Hoehnke radical (H-radical)* ϱ is an ideal-mapping satisfying:

(i) S is a nearring and t is a homomorphism of S implies $t(\varrho(S)) \subseteq \varrho(t(S))$;

(ii) S is a nearring implies $\varrho(S/\varrho(S)) = \{0\}$.

A H-radical ϱ satisfying, $\varrho(\varrho(S)) = \varrho(S)$ for all nearrings S, is called *idempotent*.
A H-radical ϱ satisfying, $\varrho(K) = K$ implies $K \subseteq \varrho(S)$ for all ideals K of a nearring S, is called *complete*.
A complete, idempotent H-radical is a *Kurosh-Amitsur radical*.
A H-radical ϱ is *ideal-hereditary* if S is a nearring and J is an ideal of S implies $\varrho(J) = J \cap \varrho(S)$.

2 S-groups of type 2^* and the J_{2^*} radical

In this section, we only consider zerosymmetric nearring. S denotes a zerosymmetric nearring and Γ a (left) S-group.
Consider Γ, which is an S-group of type 2 . For $\gamma \in \Gamma$, $S\gamma$ is an S-subgroup of Γ. So either $S\gamma = \{0\}$ or Γ. We define $\Gamma^0 := \{\gamma \in \Gamma \mid S\gamma = \{0\}\}$ and $\Gamma^1 := \{\gamma \in \Gamma \mid S\gamma = \Gamma\}$. We have $\Gamma = \Gamma^0 \cup \Gamma^1$ and $\Gamma^0 \cap \Gamma^1 = \emptyset$. Also, Γ^0 does not contain a subgroup of Γ as Γ is an S-group of type 2.

Definition 2.1. *Let Γ be a type-2, S-group. Then Γ is a* type 2^*, *S-group if $\gamma_1, \gamma_2 \in \Gamma^1$ and $s\gamma_1 = s\gamma_2$ for all $s \in S$ implies $\gamma_1 = \gamma_2$.*

Lemma 2.2. *Suppose that K is an ideal of S and Γ is a type 2^*, K-group. Then Γ is a type 2^*, S-group.*

Proof. K is an ideal of S and Γ is a type 2^*, K-group. So Γ is a type 2, K-group and it has a generator γ_0. It is clear that $h : K \to \Gamma$ defined by $h(k) = k\gamma_0, k \in K$, is a $K-$epimorphism with kernel $M := (0 : \gamma_0)_K$. So K/M is K-isomorphic to Γ. Now, K/M is a type 2, K-group and by Theorem 3.34 of [4], K/M is a type 2, S-group, where $s(k + M) = sk + M$. From the proof of the theorem, for $x + M \in K/M$, $S(x + M) = M$ implies $K(x + M) = M$. So, for $x + M \in K/M$, $K(x + M) = M$ if and only if $S(x + M) = M$ and hence $x + M$ is a generator of the K-group K/M if and only if it act as a generator of S-group K/M. Let $x_1 + M, x_2 + M$ be generators of the S-group K/M with $t(x_1 + M) = t(x_2 + M)$ for all $t \in S$. Now $x_1 + M, x_2 + M$ are generators of the K-group K/M and $p(x_1 + M) = p(x_2 + M)$ for all $p \in K$. Since K/M is a type 2^*, K-group, $x_1 + M = x_2 + M$. Therefore, K/M is a type 2^*, S-group. Hence Γ is a type 2^*, S-group. ∎

Remark 2.3. *In Lemma 2.2, g given by $g(p+M) = p\gamma_0$ is a K-isomorphism from K/M onto Γ and K/M is an S-group with $s(k + M) = sk + M, s \in S$.*

Let $\gamma \in \Gamma$. We have $\gamma = k\gamma_0$, for some $k \in K$. Define $s\gamma = s(k\gamma_0) = s(g(k+M)) := g(sk+M)$. Since g is a K-isomorphism, this action of S on Γ makes Γ an S-group. Moreover g is an S-isomorphism of K/M onto Γ. Since K/M is a type 2^*, S-group, Γ is a type 2^*, S-group. In addition, the restricted action of S to K on Γ is same as the existing one of K on Γ.

Definition 2.4. *Let S/M be a type 2^*, S-group, M is a modular left ideal of S. Then M is called a* 2^*-modular left ideal *of S.*

Definition 2.5. *An ideal K of S is* 2^*-primitive *if $K = (0:\Gamma) := \{s \in S \mid s\gamma = \{0\}\}$ for some S-group Γ of type 2^*.* S is a 2^*-primitive nearring *if the zero ideal of S is 2^*-primitive.*

Definition 2.6. The (left) Jacobson radical of type 2^* of S *is the intersection of all 2^*-primitive ideals of S. It is denoted by $J_{2^*}(S)$.*

Remark 2.7. $J_{2^*}(S) = \cap\{Q \mid Q$ *is a 2^* primitive ideal of* $S\} = \cap\{(0:\Gamma) \mid \Gamma$ *is an S-group of type*$2^*\} = \cap\{M \mid M$ *is a 2^*-modular left ideal of* $S\}$.

Theorem 2.8. $J_{2^*}(S) \cap K \subseteq J_{2^*}(K)$, *for any ideal K of S.*

Proof. We have that K is an ideal of S. If there is no K-group of type 2^*, then $J_{2^*}(S) \cap K \subseteq K = J_{2^*}(K)$. Suppose that Γ is a K-group of type 2^*. By Lemma 2.2, Γ is a type 2^*, S-group, with $(0:\Gamma)_S \cap K = (0:\Gamma)_K$. Therefore, $J_{2^*}(S) \cap K \subseteq J_{2^*}(K)$. □

Remark 2.9. *J_{2^*} is the H-radical in zerosymmetric nearrings determined by the class of all 2^*-primitive nearrings.*

Corollary 2.10. *J_{2^*} is an idempotent radical in zerosymmetric nearrings.*

Proof. Let $K := J_{2^*}(S)$. By Theorem 2.8, $J_{2^*}(S) = J_{2^*}(S) \cap J_{2^*}(S) \subseteq J_{2^*}(J_{2^*}(S))$. Obviously, $J_{2^*}(J_{2^*}(S)) \subseteq J_{2^*}(S)$. Therefore, $J_{2^*}(J_{2^*}(S)) = J_{2^*}(S)$ and hence J_{2^*} is an idempotent radical. □

Lemma 2.11. *Suppose that K is an ideal of S and Γ is an S-group of type 2^* with $K\Gamma \neq \{0\}$. Then Γ is a K-group of type 2^*.*

Proof. We have that K is an ideal of S and Γ is a type 2^*, S-group and $K\Gamma \neq \{0\}$. Let $\Delta \neq \{0\}$ be a K-subgroup of Γ. We have $K\Delta \subseteq \Delta$. Suppose that $K\Delta = \{0\}$. Now, $S\Delta \neq \{0\}$. We get $\delta \in \Delta$ such that $S\delta \neq \{0\}$. So $S\delta = \Gamma$. Now, $K\Gamma = K(S\delta) = (KS)\delta \subseteq K\delta = \{0\}$, a contradiction. So $K\Delta \neq \{0\}$. Hence $\delta_0 \in \Delta$ such that $K\delta_0 \neq \{0\}$. Since $S(K\delta_0) = (SK)\delta_0 \subseteq$

$K\delta_0$, $K\delta_0 = \Gamma$ and $\Delta = \Gamma$. Therefore, Γ is a K-group of type 2. Let $\delta_1, \delta_2 \in \Gamma$ and $K\delta_1 = \Gamma = K\delta_2$. It is clear that $S\delta_1 = \Gamma = S\delta_2$. Suppose that $k\delta_1 = k\delta_2$ for all $k \in K$. We claim that $(0 : \delta_1)_S$ and $(0 : \delta_2)_S$ are equal. On the contrary suppose that are different. Since $S/(0 : \delta_i)_S$ is S-isomorphic to Γ, $i = 1, 2$, we have $S = (0 : \delta_2)_S + (0 : \delta_1)_S$. We get $e_1, e_2 \in K$ such that $e_1\delta_1 = \delta_1$ and $e_2\delta_1 = \delta_2$. Now, $Ke_1 + (0 : \delta_1)_K = K = Ke_2e_1 + (0 : \delta_1)_K$. Let $c \in K \subseteq S = (0 : \delta_1)_S + (0 : \delta_2)_S$. Now, $c = a + b, a \in (0 : \delta_1)_S, b \in (0 : \delta_2)_S$. Let $x \in K$. $xc = x(a + b) = (x(a + b) - xb) + xb \in ((0 : \delta_1)_S \cap K) + ((0 : \delta_2)_S \cap K) = (0 : \delta_1)_K$ as $k\delta_1 = k\delta_2$, for all $k \in K$. Therefore, $K^2 \subseteq (0 : \delta_1)_K$. We have $e_1 = ye_1 + d, y \in K, d \in (0 : \delta_1)_K$ as $K = Ke_1 + (0 : \delta_1)_K$. Also, $xe_1 = x(ye_1+d) = (x(ye_1+d)-xye_1)+xye_1 \in (0 : \delta_1)_K+(0 : \delta_1)_K = (0 : \delta_1)_K$, as $K^2 \subseteq (0 : \delta_1)_K$. Therefore, $K = Ke_1 + (0 : \delta_1)_K \subseteq (0 : \delta_1)_K + (0 : \delta_1)_K = (0 : \delta_1)_K$, a contradiction. So $(0 : \delta_1)_S$ and $(0 : \delta_2)_S$ are equal. Let $s \in S$. Now, $s - se_1 \in (0 : \delta_2)_S$. So $s\delta_2 = (se_1)\delta_2$. Moreover, $(se_1)\delta_1 = (se_1)\delta_2$. For all $s \in S, s\delta_1 = (se_1)\delta_1 = (se_1)\delta_2 = s\delta_2$. Therefore, $\delta_1 = \delta_2$ and hence Γ is a type 2^* K-group. □

Theorem 2.12. *If K is an ideal of S, then $J_{2^*}(K) \subseteq K \cap J_{2^*}(S)$.*

Proof. We have that K is an ideal of S. Suppose there is no S-group of type 2^* . Clearly, $J_{2^*}(S) = S$ and $J_{2^*}(K) \subseteq K \cap J_{2^*}(S)$. Let Γ be an S-group of type 2^* . If $K \subseteq (0 : \Gamma)_S$, then $K = K \cap (0 : \Gamma)_S$. So assume that $K \not\subseteq (0 : \Gamma)_S$. By Lemma 2.11, Γ is a type 2^*, K-group and $K \cap (0 : \Gamma)_S = (0 : \Gamma)_K$. Therefore, $J_{2^*}(K) \subseteq K \cap J_{2^*}(S)$. □

Corollary 2.13. *J_{2^*} is a complete radical in zerosymmetric nearrings.*

Proof. Let K be an ideal of S and $J_{2^*}(K) = K$. By Theorem 2.12, $K = J_{2^*}(K) \subseteq K \cap J_{2^*}(S)$ and that $K \subseteq J_{2^*}(S)$. So J_{2^*} is a complete radical. □

Let K be an ideal of S. From Theorems 2.8 and 2.12 it follows that $J_{2^*}(K) = K \cap J_{2^*}(S)$. So we have:

Theorem 2.14. *J_{2^*} is ideal-hereditary in the class of zerosymmetric nearrings.*

Corollary 2.15. *J_{2^*} is a Kurosh-Amitsur radical in zerosymmetric nearrings.*

We now present a type 2^*, S-group that is not a type3, S-group. This provides a nearring which a 2^*-primitive nearring but not a 3-primitive nearring.

Example 2.16. *Let* $(\Gamma, +)$ *be group of order greater than* 2*. Consider the nearring* $M_0(\Gamma)$*. Let* $0 \in \Delta$ *be a subset of* Γ *containing no non-zero subgroup of* Γ *with* $\mid \Delta \mid \geq 2$*. Now,* $S := \{s \in M_0(\Gamma) \mid s(\gamma) = 0$ *for all* $\gamma \in \Delta\}$ *is a subnearring of* $M_0(\Gamma)$*.* S *is a zerosymmetric nearring and* Γ *is an* S*-group of type* 2^* *. Since* $(0 : \Gamma)_S = \{0\}$*,* S *is a* 2^**-primitive nearring. Clearly,* Γ *is not a* S*-group of type* 3*. It can be easily verified that* S *is not an equiprime nearring. So, by Lemma 4.1 of [1],* S *is not a* 3*-primitive nearring. Moreover,* Γ *is not an* S*-group of type* $5/2$ *.*

Remark 2.17. *In Example 2.16, if* Γ *is a finite group, then, by Theorem 4.46 of [4],* S *is a simple nearring. This shows that we have simple nearrings which are* J_{2^*}*-semisimple and* J_3*-radical nearrings.*

Example 2.18. *Suppose that* $(S, +)$ *is a group of order* $p > 3$*, where* p *is a prime number. Let* $0 \in A$ *be a subset of* S *with* $2 \leq\mid A \mid\leq p - 2$*. Define a product on* S *by* $s_1.s_2 = s_1$ *if* $s_2 \notin A$ *and* $s_1.s_2 = 0$ *if* $s_2 \in A$*,* $s_1, s_2 \in S$*.* S *is a zerosymmetric simple nearring. Moreover,* S *is a type* 2*,* S*-group and any type* 2*,* S*-group is* S*-isomorphic to* S*. So* S *is a* 2*-primitive nearring. It is clear that* S *is not a type* 2^**,* S*-group and that there is no type* 2^**,* S*-group. Hence* S *is* J_2*-semisimple and* J_{2^*}*-radical nearring. Note that if* $A = \{0\}$*, then* S *is a type* $5/2$*,* S*-group and that* S *is a simple* $5/2$*-primitive nearring which is a* J_{2^*}*-radical nearring as* S *is not an* S*-group of type* 2^**.*

Remark 2.19. *Note that an* S*-group type* 2^* *has the basic characteristics of an* S*-group of type* 3*. At the same time, as seen in example 2.16, some natural nearrings are retained as* J_{2^*}*-semisimple nearrings even though they are* J_3*-radical nearrings.*

References

[1] G.L. Booth, N.J. Groenewald, S. Veldsman, *A Kurosh-Amitsur prime radical for near-rings*, Comm. Algebra, **18,** (1990), no. 9, 3111–3122.

[2] W.M.L. Holcombe, *A hereditary radical for near-rings*, Studia Sci. Math. Hungar., **17,** (1982), 453–456.

[3] K. Kaarli, *Radicals in near-rings*, Tartu Rikkl. Ul. Toitmetised, 390, (1976), 134–171 (in Russian).

[4] G. Pilz, *Near-rings*, revised ed., North-Holland, Amsterdam, 1983.

[5] R. Srinivasa Rao, K. Siva Prasad, *A kurosh-Amitsur left Jacobson radical for right near-rings*, Bull. Korean Math. Soc., **45,** (2008), no. 3, 457–466.

[6] S. Veldsman, *On the non-hereditariness of radical and semisimple classes of near-rings*, Studia Sci. Math. Hungar., **24,** (1989), 315–323.