International Journal of Mathematics and
Computer Science, **19**(2024), no. 3, 635–642

Coefficient Bounds of a class of Bi-Univalent Functions Related to Gegenbauer Polynomials

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(Received November 6, 2023, Accepted December 11, 2023,
Published February 12, 2024)

Abstract

In this paper, we introduce and investigate a class of bi-univalent functions associated with Gegenbauer Polynomials. For functions in this class, we derive the estimations for the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Moreover, we obtain the classical Fekete-Szegö inequality of functions belonging to this class.

1 Introduction

Let $\mathcal{A}$ be the family of all analytic functions f that are defined on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Any function $f \in \mathcal{A}$ has the following Taylor-Maclarin series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \text{where} \quad z \in \mathbb{D}. \tag{1.1}$$

Let $\mathcal{S}$ denote the class of all functions $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$. Let the functions f and g be analytic in $\mathbb{D}$. We say the function f is subordinate

Key words and phrases: Univalent and Bi-Univalent Functions, Principle of Subordination, Gegenbauer Polynomials, Chebyshev polynomials, Coefficient estimates, Fekete-Szegö Inequality.
AMS (MOS) Subject Classifications: 30C45, 30C50, 33C45, 33C05, 11B39.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

by the function g in $\mathbb{D}$, denoted by $f(z) \prec g(z)$ for all $z \in \mathbb{D}$, if there exists a Schwarz function w, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$, such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. In particular, if the function g is univalent over $\mathbb{D}$, then $f(z) \prec g(z)$ equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. For more information about the Subordination Principle, we refer the readers to the monographs [7], [15], [16].

It is well-known that univalent functions are injective (one-to-one) functions. Hence, they are invertible and the inverse functions may not be defined on the entire unit disk $\mathbb{D}$. In fact, the Koebe one-quarter Theorem states that the image of $\mathbb{D}$ under any function $f \in \mathcal{S}$ contains the disk $D(0, 1/4)$ of center 0 and radius $1/4$. Accordingly, every function $f \in \mathcal{S}$ has an inverse $f^{-1} = g$ which is defined as

$$g(f(z)) = z, \quad z \in \mathbb{D}$$

$$f(g(w)) = w, \quad |w| < r(f); \quad r(f) \geq 1/4.$$

Moreover, the inverse function is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{1.2}$$

For this reason, we define the class Σ as follows:
A function $f \in \mathcal{A}$ is said to be bi-univalent if both f and f^{-1} are univalent in $\mathbb{D}$. Therefore, let Σ denote the class of all bi-univalent functions in $\mathcal{A}$ which are given by equation (1.1). For more information about univalent and bi-univalent functions we refer the readers to the articles [13], [14] and the monograph [6], [8] and the references therein.

The subject of the geometric function theory in complex analysis has been investigated by many researchers in recent years. The typical problem in this field is studying a functional made up of combinations of the initial coefficients of the functions $f \in \mathcal{A}$. For a function in the class $\mathcal{S}$, it is well-known that $|a_n|$ is bounded by n. Moreover, the coefficient bounds give information about the geometric properties of those functions. For instance, the bound for the second coefficients of the class $\mathcal{S}$ gives the growth and distortion bounds for the class. In addition, the Fekete-Szegö functional arises naturally in the investigation of univalency of analytic functions. In the year 1933, Fekete and Szegö [11] found the maximum value of $|a_3 - \lambda a_2^2|$, as a function of the real parameter $0 \leq \lambda \leq 1$ for a univalent function f. Since then, the problem of dealing with the Fekete-Szegö functional for $f \in \mathcal{A}$

with any complex λ is known as the classical Fekete-Szegö problem. Many researchers have investigated the Fekete-Szegö functional and the other coefficient estimates problems; for example, see the articles [1], [2], [4], [5], [9], [10], [11], [13], [14], [18] and the references therein.

2 Preliminaries

In this section, we present some information that are crucial for the main results of this paper. For any real numbers $\mu, t \in \mathbb{R}$, with $\mu \geq 0$ and $-1 \leq t \leq 1$, and $z \in \mathbb{D}$ the generating function of Gegenbauer polynomials is given by

$$H_\mu(z,t) = (z^2 - 2tz + 1)^{-\mu}.$$

Moreover, for any fixed t, the function $H_\mu(z,t)$ is analytic on the unit disk $\mathbb{D}$ and its Taylor-Maclaurin series is given by

$$H_\mu(z,t) = \sum_{n=0}^{\infty} C_n^\mu(t) z^n.$$

In addition, Gegenbauer polynomials can be defined in terms of the following recurrence relation:

$$C_n^\mu(t) = \frac{2t(n+\mu-1)C_{n-1}^\mu(t) - (n+2\mu-2)C_{n-1}^\mu(t)}{t}, \tag{2.3}$$

with initial values

$$C_0^\mu(t) = 1, \quad C_1^\mu(t) = 2\mu t, \quad \text{and} \quad C_2^\mu(t) = 2\mu(\mu+1)t^2 - \mu. \tag{2.4}$$

It is well-known that the Gegenbauer polynomials and their special cases such as Legendre polynomials $L_n(t)$ and the Chebyshev polynomials of the second kind $T_n(t)$ are orthogonal polynomials, where the values of μ are $\mu = 1/2$ and $\mu = 1$ respectively. More precisely, $L_n(t) = C_n^{1/2}(t)$ and $T_n(t) = C_n^1(t)$ and $T_n(t) = C_\mu^1(t)$. For more information about the Gegenbauer polynomials and their special cases, we refer the readers to the articles [3], [9], [12], [14], [17], [18], the monograph [6], [8] and the references therein.

Definition 2.1. *A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{B}_\Sigma(\mu,t)$ if it satisfies the following subordination conditions, associated with the Gegenbauer Polynomials, for all z, $w \in \mathbb{D}$:*

$$\left(1+\frac{zf''(z)}{f'(z)}\right)\left(\frac{zf'(z)}{f(z)}\right)^{-1} \prec H_\mu(z,t) \tag{2.5}$$

and

$$\left(1+\frac{wg''(w)}{g'(w)}\right)\left(\frac{wg'(w)}{g(w)}\right)^{-1} \prec H_\mu(w,t), \tag{2.6}$$

where $\mu > 0$, $t \in [-1,1]$ and $g(w)$ is defined by equation (1.2).

The following lemma is a well-known fact and it is crucial for this paper's results.

Lemma 2.2. *[10] Let k, $l \in \mathbb{R}$ and p, $q \in \mathbb{C}$. If $|p| < r$ and $|q| < r$,*

$$|(k+l)p+(k-l)q| \leq \begin{cases} 2|k|r, & \text{if } |k| \geq |l| \\ 2|l|r, & \text{if } |k| \leq |l| \end{cases}$$

In this paper, we investigate a subclass of bi-univalent functions Σ in the open unit disk $\mathbb{D}$, which we denote by $\mathcal{B}_\Sigma(\mu,t)$. For functions in this subclass, we obtain the estimates for the initial Taylor-Maclarin coefficients $|a_2|$ and $|a_3|$. Furthermore, we examine the corresponding Fekete-Szegö functional problem for functions in this class.

3 Coefficient estimates of the class $\mathcal{B}_\Sigma(\mu,t)$

In this section, we provide bounds for the initial Taylor-Maclaurin coefficients for functions belonging to the class $\mathcal{B}_\Sigma(\mu,t)$ which are given by equation (1.1).

Theorem 3.1. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma(\mu,t)$. Then*

$$|a_2| \leq \frac{2\sqrt{2}\mu|t|^{3/2}}{\sqrt{|(1+\mu)t^2-1|}}, \tag{3.7}$$

and

$$|a_3| \leq \frac{8\mu^2t^2+\mu|t|}{2} \tag{3.8}$$

Proof. Let f be in the class $\mathcal{B}_\Sigma(\mu,t)$. Then, using (2.5) and (2.6), we can find two analytic functions p and q on the unit disk $\mathbb{D}$ such that

$$\left(1+\frac{zf''(z)}{f'(z)}\right)\left(\frac{zf'(z)}{f(z)}\right)^{-1} \prec H_\mu(t,p(z)), \tag{3.9}$$

and

$$\left(1+\frac{wg''(w)}{g'(w)}\right)\left(\frac{wg'(w)}{g(w)}\right)^{-1} \prec H_\mu(t,q(w)), \tag{3.10}$$

where $p(z)=\sum_{n=1}^{\infty} p_n z^n$ and $q(w)=\sum_{n=1}^{\infty} q_n w^n$ for all $z,w\in\mathbb{D}$. Also, for all $z,w\in\mathbb{D}$, $|p(z)|<1$ and $|q(z)|<1$. Moreover, it is well-known that [6] for all $j\in\mathbb{N}$, $|p_j|\le 1$ and $|q_j|\le 1$.
Now, upon comparing the corresponding coefficients in both sides of (3.9) and (3.10), we obtain the following:

$$a_2 = C_1^\mu(t)p_1, \tag{3.11}$$

$$4(a_3-a_2^2) = C_1^\mu(t)p_2 + C_2^\mu(t)p_1^2, \tag{3.12}$$

$$-a_2 = C_1^\mu(t)q_1, \tag{3.13}$$

and

$$-4(a_3-a_2^2) = C_1^\mu(t)q_2 + C_2^\mu(t)q_1^2 \tag{3.14}$$

Using equations (3.11) and (3.13), we get

$$p_1 = -q_1, \tag{3.15}$$

and

$$2a_2^2 = [C_1^\mu(t)]^2(p_1^2+q_1^2) \tag{3.16}$$

Moreover, adding equations (3.12) and (3.14), we get

$$0 = (p_2+q_2)C_1^\mu(t) + (p_1^2+q_1^2)C_2^\mu(t) \tag{3.17}$$

Substituting the value of $(p_1^2+q_1^2)$ from equation (3.17) in the right hand-side of equation (3.16), we get

$$a_2^2 = \frac{-[C_1^\mu(t)]^3(p_2+q_2)}{2C_2^\mu(t)} \tag{3.18}$$

By taking $C_1^\mu(t)=2\mu t$ and $C_2^\mu(t)=2\mu(\mu+1)t^2-\mu$ in equation (3.18), it further yields

$$a_2^2 = \frac{-4\mu^3t^3(p_2+q_2)}{(\mu+1)t^2-1}.$$

Using the facts $|p_2| \leq 1$ and $|q_2| \leq 1$, we get the desired estimate of a_2. Now, we look for the bound of $|a_3|$. Subtracting equation (3.14) from equation (3.12) and using equation (3.15), we get

$$a_3 = \frac{C_1^\mu(t)}{8}(p_2 - q_2) + a_2^2. \tag{3.19}$$

Using equation (3.16), we obtain

$$a_3 = \frac{[C_1^\mu(t)]^2}{2}(p_1^2 + q_1^2) + \frac{C_1^\mu(t)}{8}(p_2 - q_2).$$

Hence, using equation (2.4) and the facts $|p_j| \leq 1$ and $|q_j| \leq 1$ for all $j \in \mathbb{N}$, we get the desired estimate of a_3. This completes the proof. □

The following corollary is a consequence of Theorem 3.1 when taking $\mu = 1$. These initial coefficient estimates are related to Chebyshev polynomials of the second kind.

Corollary 3.2. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma(\mu, t)$. Then*

$$|a_2| \leq \frac{2\sqrt{2}|t|^{3/2}}{\sqrt{|2t^2 - 1|}}, \tag{3.20}$$

and

$$|a_3| \leq \frac{8t^2 + |t|}{2}. \tag{3.21}$$

4 Fekete-Szegö functional of the class $\mathcal{B}_\Sigma(\mu, t)$

In this section, we consider the classical Fekete-Szegö functional of functions belonging to the class $\mathcal{B}_\Sigma(\mu, t)$.

Theorem 4.1. *Let the function f given by (1.1) be in the class $\mathcal{B}_\Sigma(\mu, t)$. Then, for some $\lambda \in \mathbb{R}$,*

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{\mu|t|}{2}, & if \ \ |\lambda - 1| \leq \frac{|(\mu+1)t^2-1|}{16\mu t^2} \\ \frac{8\mu^2 t^2 |t\lambda - t|}{|(\mu+1)t^2-1|}, & if \ \ |\lambda - 1| \geq \frac{|(\mu+1)t^2-1|}{16\mu t^2}, \end{cases} \tag{4.22}$$

Proof. For some real number λ, using equation (3.19) we have

$$a_3 - \lambda a_2^2 = \frac{C_1^\mu(t)}{8}(p_2 - q_2) + (1 - \lambda)a_2^2.$$

In view of equation (3.18), we obtain

$$\begin{aligned} a_3 - \lambda a_2^2 =& \frac{C_1^{\mu}(t)(p_2 - q_2)}{8} + \frac{(\lambda-1)(p_2+q_2)[C_1^{\mu}(t)]^3}{2C_2^{\mu}(t)} \\ =& C_1^{\mu}(t)\left\{\left(\frac{(\lambda-1)[C_1^{\mu}(t)]^2}{2C_2^{\mu}(t)} + \frac{1}{8}\right)p_2 + \left(\frac{(\lambda-1)[C_1^{\mu}(t)]^2}{2C_2^{\mu}(t)} - \frac{1}{8}\right)q_2\right\} \end{aligned}$$

Therefore, using Lemma 2.2 and the initial values (2.4), we get the desired inequality (4.22), which completes the proof. □

The following corollary is just a consequence of Theorem 4.1. Taking $\mu = 1$, we get the Fekete-Szegö inequality that is related to Chebyshev polynomials of the second kind.

Corollary 4.2. *Let the function f given by (1.1) be in the class $\mathcal{B}_\Sigma(\mu, t)$. Then for some $\lambda \in \mathbb{R}$,*

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{|t|}{2}, & \text{if } \; |\lambda - 1| \leq \frac{|2t^2-1|}{16t^2} \\ \frac{8t^2|t\lambda - t|}{|2t^2-1|}, & \text{if } \; |\lambda - 1| \geq \frac{|2t^2-1|}{16t^2}. \end{cases} \tag{4.23}$$

References

[1] W. Al-Rawashdeh, Fekete-Szegö Functional of a Subclass of Bi-Univalent Functions Associated with Gegenbauer Polynomials, preprint.

[2] W. Al-Rawashdeh, Horadam Polynomials and a Class of Bi-Univalent Functions Defined by Ruscheweyh Operator, preprint.

[3] W. Al-Rawashdeh, Applications of Gegenbauer Polynomials on a Class of Non-Bazilevic Functions, preprint.

[4] M. Cağlar, H. Orhan, M. Kamali, Fekete-Szegö problem for a subclass of analytic functions associated with Chebyshev polynomials, *Bol. Soc. Paran. Mat.*, **40,** (2022), 1–6.

[5] J. H. Choi, Y. C. Kim, T. Sugawa, A general approach to the Fekete-Szegö problem, *Journal of the Mathematical Society of Japan*, **59,** (2007), 707–727.

[6] P. Duren, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften, **259,** Springer-Verlag, New York, 1983.

[7] P. Duren, Subordination in Complex Analysis, *Lecture Notes in Mathematics*, Springer, Berlin, Germany, **599,** (1977), 22–29.

[8] A. W. Goodman, *Univalent Functions*, 2 volumes, Mariner Publishing Co. Inc., 1983.

[9] M. Kamali, M. Cağlar, E. Deniz, M. Turabaev, Fekete Szegö problem for a new subclass of analytic functions satisfying subordinate condition associated with Chebyshev polynomials, *Turkish J. Math.*, **45,** (2021), 1195–1208.

[10] F. R. Keogh, E.P. Merkes, A Coefficient inequality for certain classes of analytic functions, *Proceedings of the American Mathematical Society*, **20,** (1969), 8–12.

[11] M. Fekete, G. Szegö, Eine Bemerkung Über ungerade Schlichte Funktionen, *Journal of London Mathematical Society*, **s1-8,** (1933), 85–89.

[12] K. Kiepiela, I. Naraniecka, J. Szynal, The Gegenbauer polynomials and typically real functions, *Journal of Computational and Applied Mathematics*, **153,** nos. 1-2, (2003), 273–282.

[13] M. Lewin, On a coefficient problem for bi-univalent functions, *Proceedings of the American Mathematical Society*, **18,** no. 1, (1967), 63–68.

[14] N. Magesh, S. Bulut, Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functions related to pseudo-starlike functions, *Afrika Matematika*, **29,** nos. 1–2, (2018), 203–209.

[15] S. Miller, P. Mocabu, *Differential Subordination: Theory and Applications*, CRC Press, New York, 2000.

[16] Z. Nehari, *Conformal Mappings*, McGraw-Hill, New York, 1952.

[17] H. Orhan, N. Magesh, V. Balaji, Second Hankel determinant for certain class of bi-univalent functions defined by Chebyshev polynomials, *Asian-European Journal of Mathematics*, **12,** no. 2, (2019), 1950017.

[18] H. M. Srivastava, M. Kamali, A. Urdaletova, A study of the Fekete-Szegö functional and coefficient estimates for subclasses of analytic functions satisfying a certain subordination conditionv and associated with the Gegenbauer polynomials, *AIMS Mathematics*, **7,** no. 2, (2021), 2568–2584.