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Assessing the Efficiency of Several Approaches for Estimating the Two Unknown Parameters of the Rayleigh Distribution

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Abstract

The main objective of this paper is to compare the efficiency of different methods involving our proposed method in estimating the two unknown parameters of the Rayleigh distribution. For this purpose, the Monte Carlo simulation analysis is performed. The methods of estimation that we have considered are the maximum likelihood, moments, ordinary least squares and our proposed relative least squares method. Two indicators are used to assess the efficiency for each method; namely, the total deviation (TD) and the mean square error

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(MSE). The smaller the value of these indicators, the more efficient is the method.

1 Introduction

The Rayleigh distribution was first proposed by Lord Rayleigh in 1880 [1] in the field of acoustics and is one of the most important distributions used with Skewed Data. It is closely related to other commonly used distributions such as the Weibull distribution, the Chi-Square distribution and the extreme value distribution. The Rayleigh distribution has been widely used by engineers and physicists to model radiation, radar images and other related phenomena [2]. The authors in [3], [4], and [5] study the minimax shrinkage estimator technique, Double Stage Shrinkage-Bayesian Estimator and Inequalities system, respectively.
The rest of this paper is organized as follows:
In Section 2, we include a quick review of some of the basic concepts related to the Rayleigh distribution such as the density function, the distribution function and the Moments about origin through which we can find the mean and variance of the distribution.
In Section 3, we include the theoretical study of four of the estimation methods used in this paper. More specifically, maximum likelihood method (MLM), moments method (MM), ordinary least squares method (OLSM) and our proposed relative least squares method (RLSM).
In Section 4, we include the experimental aspect represented by conducting a Monte Carlo simulation analysis to compare the efficiency of the estimators under study by adopting different sample sizes and different initial values for the true parameters.

2 On The Rayleigh distribution

The probability density function for the continuous random variable X that follows the Rayleigh distribution is [5]:

$$f(x,\alpha,\beta) = \frac{(x-\alpha)}{\beta^2} exp\left[\frac{-(x-\alpha)^2}{2\beta^2}\right] \ , \ \alpha < x < \infty, \ \beta > 0 \tag{2.1}$$

Accordingly, the distribution function can be determined as:

$$F(x,\alpha,\beta)=\int_{\alpha}^{X} f(u,\alpha,\beta)du = 1-\exp\left[\frac{-(x-\alpha)^2}{2\beta^2}\right],\ \alpha<x<\infty, \beta>0 \tag{2.2}$$

The rth moment about origin denoted by M_r can be obtained as:

$$M_r = E(x^r) = \int_{\alpha}^{\infty} x^r f(x,\alpha,\beta)dx$$

For the Rayleigh distribution, this implies that:

$$M_r = \sum_{i=0}^{r} C_i^r \alpha^i (2\beta^2)^{\frac{r-i}{2}}\ \Gamma(\frac{r-i}{2}+1) \tag{2.3}$$

Consequently, the mean and variance of the distribution can be easily obtained as follows:

$$M_x = E(x) = M_1 = \alpha + \beta\sqrt{\frac{\pi}{2}} \tag{2.4}$$

$$\sigma_x^2 = var(x) = M_2 - M_1^2 = \beta^2(2-\frac{\pi}{2}) \tag{2.5}$$

3 Estimation Methods

3.1 Maximum Likelihood Method

Assuming that $x_1, x_2, \ldots, x_n$ is a random sample from $Ray(\alpha,\beta)$, the likelihood function of this sample is given by:

$$L(\alpha,\beta;x_1,x_2,\ldots,x_n) = \prod_{i=1}^{n} f(x_i,\alpha,\beta) = \frac{1}{\beta^{2n}} e^{\frac{-1}{2\beta^2}\sum_{i=1}^{n}(x_i-\alpha)^2} \prod_{i=1}^{n}(x_i-\alpha) \tag{3.6}$$

Accordingly, the log likelihood function is:

$$\ln(\alpha,\beta;x_1,x_2,\ldots,x_n) = -2n\ln\beta - \frac{1}{2\beta^2}\sum_{i=1}^{n}(x_i-\alpha)^2 + \sum_{i=0}^{n}\ln(x_i-\alpha) \tag{3.7}$$

It is clear that the value of α that maximize $L(\alpha,\beta;x_1,\ldots,x_n)$ is the smallest value of the sample denoted by $x_{(1)}$ that is:

$$\alpha_{MLE}^{\wedge} = x_{(1)} = \min(x_1,x_2,\ldots,x_n) \tag{3.8}$$

Differentiating $\ln(\alpha, \beta; x_1, x_2, \ldots, x_n)$ in equation (3.7) with respect to β, equating the resulting derivative to zero and solving for β, we obtain the maximum likelihood estimator for β given by:

$$\beta^{\wedge}_{MLE} = \sqrt{\frac{\sum_{i=1}^{n} (x_i - x_{(1)})^2}{2n}} \tag{3.9}$$

3.2 The Moments Method

Let us assume that $\ddot{X}$ and s^2 denote the mean and variance of a random sample from $Ray(\alpha, \beta)$. Then by equating $\ddot{X}$ with the right hand side of equation (2.4), we get:

$$\ddot{X} = \alpha + \beta\sqrt{\frac{\pi}{2}} \tag{3.10}$$

Similarly, equating s^2 with the right hand side of equation (2.5) we get:

$$s^2 = \beta^2(2 - \frac{\pi}{2}) \tag{3.11}$$

Solving equation (3.10) and (3.11) for α and β we obtain the Moments estimators given as follows:

$$\alpha^{\wedge}_{ME} = s\sqrt{\frac{2}{4-\pi}} \tag{3.12}$$

$$\alpha^{\wedge}_{ME} = \ddot{X} - s\sqrt{\frac{\pi}{4-\pi}} \tag{3.13}$$

3.3 Ordinary Least Squares Method

Let us assume that $x_1, x_2, \ldots, x_n$ is a random simple from $Ray(\alpha, \beta)$.
Putting $u_i = F(x_i), i = 1, 2, \ldots, n$ where $F(x_i)$ is the distribution function defined in equation (2.2), then:

$$u_i = 1 - e^{-\frac{1}{2\beta^2}(x_i-\alpha)^2},$$

which implies that:

$$-\frac{1}{2\beta^2}(x_i - \alpha)^2 = \ln(1 - u_i)$$

Solving for x_i, we get:

$$x_i = \alpha + \beta[-2\ln(1 - u_i)]^{0.5} \tag{3.14}$$

Putting:

$$z_i = [-2\ln(1-u_i)]^{\frac{1}{2}}, \quad i = 1, 2, \ldots, n, \tag{3.15}$$

the ordinary least squares estimators for α and β can be obtained by solving the following two equations [6]:

$$\sum_{i=1}^{n} x_i = n\alpha + \beta \sum_{i=1}^{n} z_i,$$

$$\sum_{i=1}^{n} x_i z_i = \alpha \sum_{i=1}^{n} z_i + \beta \sum_{i=1}^{n} z_i^2$$

As a result,

$$\alpha^{\wedge}_{OLS} = \frac{\sum_{i=1}^{n} z_i^2 \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} z_i \sum_{i=1}^{n} z_i \quad x_i}{n \sum_{i=1}^{n} z_i^2 - (\sum_{i=1}^{n} z_i \quad)^2}. \tag{3.16}$$

$$\beta^{\wedge}_{OLS} = \frac{n \sum_{i=1}^{n} z_i x_i - (\sum_{i=1}^{n} z_i)(\sum_{i=1}^{n} x_i)}{n \sum_{i=1}^{n} z_i^2 - (\sum_{i=1}^{n} z_i)^2}. \tag{3.17}$$

3.4 Relative Least Squares Method (Proposed method)

The relative least squares (RLS) estimators can be obtained by minimizing the residuals relative to the observed values of dependent variable [7].
In this paper, using the RLS method, we estimate the two unknown parameters α and β of the Rayleigh distribution. Assume

$$T = \sum_{i=1}^{n} [\frac{1}{x_i}(x_i - \alpha - \beta z_i)]^2.$$

Represent the relative residuals sum of squares, where z_i is given in equation (3.8). Moreover, if we let $w_i = \frac{1}{x_i}$ and $t_i = \frac{z_i}{x_i}$, $i = 1, 2, \ldots, n$, then

$$T = \sum_{i=1}^{n} (1 - \alpha w_i - \beta t_i)^2.$$

Differentiating T with respect to α and β respectively, equating the resulting derivative to zero, and rearranging the terms we get:

$$\sum_{i=1}^{n} w_i = \alpha \sum_{i=1}^{n} {w_i}^2 + \beta \sum_{i=1}^{n} w_i t_i$$

$$\sum_{i=1}^{n} t_i = \alpha \sum_{i=1}^{n} w_i t_i + \beta \sum_{i=1}^{n} {t_i}^2$$

Solving for α and β, we obtain the (RLS) estimators given by:

$$\alpha^{\wedge}_{RLS} = \frac{\sum_{i=1}^{n} w_i \quad t_i \sum_{i=1}^{n} t_i - \sum_{i=1}^{n} w_i \sum_{i=1}^{n} t_i^2}{\left(\sum_{i=1}^{n} w_i \quad t_i\right)^2 - \sum_{i=1}^{n} w_i^2 \sum_{i=1}^{n} t_i^2} \tag{3.18}$$

$$\beta^{\wedge}_{RLS} = \frac{\sum_{i=1}^{n} w_i \quad t_i \sum_{i=1}^{n} w_i - \sum_{i=1}^{n} w_i^2 \sum_{i=1}^{n} t_i}{\left(\sum_{i=1}^{n} w_i \quad t_i\right)^2 - \sum_{i=1}^{n} w_i^2 \sum_{i=1}^{n} t_i^2} \tag{3.19}$$

4 Numerical Study

In this section, we will perform the Monte Carlo Simulation analysis to compare the efficiency of the different estimation methods that were previously mentioned.
For this purpose, three sample sizes are assumed; namely, small $(n = 10)$, medium $(n = 50)$ and large $(n = 100)$ with postulated pairs of true values $(\alpha, \beta) = (1, 1), (1, 2), (2, 1), (2, 2)$. The results are based on 1000 simulation runs. The random samples of different sizes are generated by observing that if the random variable u is distributed uniformly on $(0, 1)$, then $x = \alpha + \beta[-2\ln(1 - u)]^{\frac{1}{2}}$, a distribution denoted as $Ray(\alpha, \beta)$.

Such generated data have been employed to obtain the required estimators for the unknown parameters α and β. Two indicators are used to compare such estimators:

i The total deviation (TD) defined as [8], [9], [10]

$$TD = \left|\frac{\alpha^{\wedge} - \alpha)}{\alpha}\right| + \left|\frac{\beta^{\wedge} - \beta)}{\beta}\right|, \tag{4.20}$$

where $\alpha^{\wedge}$ and $\beta^{\wedge}$ are the estimated values of α and β, respectively.

ii The mean square error (MSE) defined as:

$$MSE = \frac{1}{n}\sum_{i-1}^{n}[F^{\wedge}(x_i) - F(x_i)]^2, \tag{4.21}$$

where $F(x_i)$ is the distribution function defined in equation (2.2) and $F^{\wedge}(x_i)$ is given as:

$$F^{\wedge}_{(x_i)} = 1 - \exp[-\frac{(x_i - \alpha^{\wedge})^2}{2\beta^{\wedge 2}}].$$

The estimator which yields the smallest (TD) and smallest (MSE) is the best.

5 Results and Conclusions

We have presented the results of thesimulation analysis that was performed to estimate the two unknown parameters of the Rayleigh distribution by using four methods of estimation and different sample sizes with different postulated values of the parameters α and β. From the comparison of the efficiencies of the different methods of estimation, we concluded that the maximum likelihood method works the best in more cases we have considered for estimating both α and β since it has the smallest total deviation and smallest mean square error, followed by the Moments method.
However, the proposed method performs well in many cases, especially when the sample size is small. These results are represented in tables 1, 2 and 3:

Table 1: Simulation analysis results at $n = 10$

Method	α	β	$\alpha^\wedge$	$\beta^\wedge$	TD	Mse
LM	1	1	1.394469	0.740035	0.654434	1.144112E-14
	1	2	1.788938	1.480070	1.048902	7.942750E-15
	2	1	2.394469	0.740035	0.457199	3.001672E-14
	2	2	2.788938	1.480070	0.654434	8.091597E-15
Moments	1	1	1.147797	1.309913	0.457710	0.000354
	1	2	1.309913	2.147797	0.383812	0.003539
	2	1	2.396663	0.969409	0.228923	0.003753
	2	2	2.309913	2.147797	0.228855	0.003539
OLSM	1	1	1.979167	0.956555	1.022612	0.010903
	1	2	1.380536	2.00731	0.384191	0.003006
	2	1	2.311602	1.045691	0.201492	0.001755
	2	2	2.411899	1.980841	0.215529	0.002699
RLS	1	1	1.947715	0.964598	0.983117	0.031585
	1	2	1.414872	1.980745	0.424499	0.015266
	2	1	2.415835	0.975236	0.232682	0.000299
	2	2	2.328526	2.043034	0.185780	0.000111

Table 2: Simulation analysis results at $n = 50$

Method	α	β	$\alpha^\wedge$	$\beta^\wedge$	TD	Mse
LM	1	1	1.174767	0.889743	0.285025	4.166021E-14
	1	2	1.349535	1.779485	0.459792	2.5350600E-14
	2	1	2.174767	0.889743	0.197641	9.108931E-14
	2	2	2.349535	1.779485	0.285025	4.166021E-14
Moments	1	1	1.396663	0.969409	0.427254	0.003753
	1	2	1.396663	1.969409	0.411959	0.003753
	2	1	2.385882	0.982723	0.210218	0.044170
	2	2	2.396663	1.969409	0.213627	0.003753
OLSM	1	1	1.362343	1.001122	0.363465	0.004712
	1	2	1.386404	1.980324	0.396242	0.004525
	2	1	2.366523	0.997573	0.185688	0.004710
	2	2	2.355805	2.007484	0.181645	0.044692
RLS	1	1	1.376641	0.989810	0.386831	0.018901
	1	2	1.367586	1.996622	0.369275	0.004709
	2	1	2.385882	0.982723	0.210218	0.044170
	2	2	2.358687	2.004582	0.181635	0.001105

Table 3: Simulation analysis results at $n = 100$

Method	α	β	$\alpha^\wedge$	$\beta^\wedge$	TD	Mse
LM	1	1	1.122719	0.923571	0.199148	7.641764E-14
	1	2	1.245438	1.847142	0.321867	4.593349E-14
	2	1	2.122719	0.923571	0.137789	1.759009E-14
	2	2	2.245438	1.847142	0.199148	7.641764E-14
Moments	1	1	1.358128	1.010033	0.368161	0.004495
	1	2	1.358128	2.010033	0.363145	0.004490
	2	1	2.358128	1.010033	0.189097	0.004495
	2	2	2.358128	2.010033	0.184080	0.004490
OLSM	1	1	1.365799	1.001811	0.367610	0.005693
	1	2	1.375747	1.993259	0.379118	0.005693
	2	1	2.390565	0.981670	0.213612	0.005580
	2	2	2.395302	1.977394	0.208954	0.005500
RLS	1	1	1.371620	0.997059	0.374561	0.008266
	1	2	1.346293	2.016486	0.354536	0.009810
	2	1	2.358186	1.007205	0.186298	0.010172
	2	2	2.368079	1.999832	0.184123	0.002718

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