International Journal of Mathematics and
Computer Science, **19**(2024), no. 4, 1137–1142

Definability of the class of SAS-flat modules

Akeel Ramadan Mehdi, Anwar Rahm Obaid

Department of Mathematics
College of Education
Al-Qadisiyah University
Al-Diwaniya, Iraq

email: akeel.mehdi@qu.edu.iq, edu-math17.14@qu.edu.iq

(Received March 11, 2024, Accepted April 22, 2024,
Published June 1, 2024)

Abstract

Let R be a ring. We study the definability of the class of SAS-flat right R-modules. Moreover, we give many properties and characterizations of the definability of this class.

1 Introduction

All modules in this paper are unital R-modules, where R stands for an associative ring with unity. We denote the category of all right (resp. left) R-modules by Mod-R (resp. R-Mod). A module N is called semiartinian if $\mathrm{soc}(N/K)$ is nonzero, for any proper submodule K of a module N. A submodule L of a module K is called small of K if $L + B = K$, where B is a submodule of K implies $B = K$ [1]. The notation $A \leq^{sas} C$ is used for a semiartinian small submodule of C. As usual, $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. A class $\mathcal{F} \subseteq$ Mod-R is called definable if it is closed under direct products, pure submodules and direct limits (see for example [2]). A pair $(\mathcal{F}, \mathcal{G})$, where $\mathcal{F} \subseteq$ Mod-R (resp. $\mathcal{G} \subseteq R$-Mod), is called almost dual pair if for any $M \in$ Mod-R, $M \in \mathcal{F} \Leftrightarrow M^* \in \mathcal{G}$ and $\mathcal{G}$ is closed under direct products and summands [2]. In module theory, injectivity and flatness for modules play important

Key words and phrases: SAS-N-flat module, SAS-N-injective module, Definable class, Semiartinian small submodule.
AMS (MOS) Subject Classifications: 16D40, 16D50, 16D10, 13C11.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

roles. Many generalizations of injectivity and flatness of modules appeared in the literature (for example see, [2], [3], [4], [5], [6], [7], [8], and [9]. In [3], the concept of SAS-K-injectivity was introduced, where a module C is called SAS-K-injective (where $K \in R$-Mod) if every left R-homomorphism from any $A \leq^{sas} K$ into C extends to K. C is called SAS-injective, when C is SAS-R-injective. In [10], the concept of SAS-N-flat (resp. SAS-flat) modules were introduced as a proper generalization of N-flat (resp. flat) module. Let $C \in$ Mod-R. Then C is called SAS-N-flat (where $N \in R$-Mod) if, for every $A \leq^{sas} N$, the sequence $0 \to C \otimes_R A \xrightarrow{I_C \otimes i_A} C \otimes_R N$ is exact. If a module C is SAS-R-flat, then C is called SAS-flat. We use SAS-N-$\mathbb{F}$ (resp. SAS-N-$\mathbb{I}$, SAS-$\mathbb{F}$, SAS-$\mathbb{I}$, $\mathbb{I}$nj, $\mathbb{P}$roj, $\mathbb{F}$lat) to denote the class of SAS-N-flat right (resp. SAS-N-injective left, SAS-flat right, SAS-injective left, injective left, projective right, flat right) R-modules. The brief symbol c.u.d.p. (resp. c.u.p.s.) stands for a class $\mathcal{F}$ means $\mathcal{F}$ is closed under direct products (resp. pure submodules).

In this paper, we study the definability of the classes SAS-N-$\mathbb{F}$ (resp. SAS-$\mathbb{F}$). Moreover, we give many characterizations and properties of the definability of these two classes. For instance, we show that if N is finitely presented, then SAS-N-$\mathbb{F}$ is a definable class $\Leftrightarrow$ R^I is SAS-N-flat, for each index I $\Leftrightarrow$ every finitely generated semiartinian small submodule of N is finitely presented $\Leftrightarrow$ (SAS-N-$\mathbb{I}$)* $\subseteq$SAS-N-$\mathbb{F}$. Furthermore, we show that if a ring R is commutative, SAS-$\mathbb{F}$ is definable over R, and SAS-$\mathbb{I}$ is c.u.p.s., then R is SAS-injective $\Leftrightarrow$ $\mathrm{Hom}_R(K, L)$ $\in$SAS-$\mathbb{I}$ for any flat (resp. projective) module L (resp. K).

2 Definability of the class of SAS-flat modules

Proposition 2.1. *The pair (SAS-N-$\mathbb{F}$,SAS-N-$\mathbb{I}$) is an almost dual pair.*

Proof. The proof follows immediately from [11, Theorem 2.3(1)] and [10, Theorem 2.3]. □

The proof of the next corollary follows directly from Proposition 2.1 and [12, Proposition 4.2.8(1,3)].

Corollary 2.2. *The class SAS-N-$\mathbb{F}$ is closed under direct limits, direct sums, pure submodules, pure extensions and pure homomorphic images.*

We can easily prove the next result.

Lemma 2.3. *Let $A, B \in R$-*Mod*. If B is projective and A is SAS-B-injective, then Ext$^1(B/K, A) = 0$, for all $K \leq^{sas} B$.*

Proposition 2.4. *Let $N \in R$-*Mod*. Consider the following conditions for SAS-N-$\mathbb{F}$.*
(1) SAS-N-$\mathbb{F}$ is a definable class.
(2) SAS-N-$\mathbb{F}$ is c.u.d.p.
(3) $R^I \in$ SAS-N-$\mathbb{F}$, for each index I.
(4) All finitely generated semiartinian small submodules of N are finitely presented.
(5) (SAS-N-$\mathbb{I}$)$^* \subseteq$SAS-N-$\mathbb{F}$.
(6) (SAS-N-$\mathbb{I}$)$^{**} \subseteq$SAS-N-$\mathbb{I}$.
(7) (SAS-N-$\mathbb{F}$)$^{**} \subseteq$SAS-N-$\mathbb{F}$.
Then (1)$\Leftrightarrow$(2)$\Leftrightarrow$(7), (5)$\Leftrightarrow$(6)$\Rightarrow$(7) and (2)$\Rightarrow$(3). If N is a finitely presented module, then (3)$\Rightarrow$(4). Moreover, the seven statements are equivalent, if N is a finitely generated free module.

Proof. (1)$\Leftrightarrow$(2)$\Leftrightarrow$(7). These follow from Proposition 2.1 and [12, Proposition 4.3.1].
(5)$\Leftrightarrow$(6). This follows from Proposition 2.1 and [12, Theorem 4.3.2].
(6)$\Rightarrow$(7). Let $U \in$ SAS-N-$\mathbb{F}$. Then [10, Theorem 2.3] implies that, $U^* \in$ SAS-N-$\mathbb{I}$. By (6), $U^{***} \in$ SAS-N-$\mathbb{I}$. By [10, Theorem 2.3], $U^{**} \in$ SAS-N-$\mathbb{F}$. Hence (SAS-N-$\mathbb{F}$)$^{**} \in$ SAS-N-$\mathbb{F}$.
(2)$\Rightarrow$(3). This is clear.
(3)$\Rightarrow$(4). Suppose that N is a finitely presented module. Let K be finitely generated with $K \leq^{sas} N$. By (3), $\prod R = R^I \in$ SAS-N-$\mathbb{F}$. By [13, Theorem 3.2.22, p. 81], K is finitely presented.
(4)$\Rightarrow$(5). Suppose that N is a finitely generated free module. Let $D \in$SAS-N-$\mathbb{I}$ and let L be finitely generated with $L \leq^{sas} N$. By Lemma 2.3, Ext$^1(N/L, D) = 0$. By (4), L is finitely presented and thus the sequence $B_2 \overset{f_2}{\to} B_1 \overset{\alpha}{\to} N \overset{\pi}{\to} N/L \to 0$ is exact, where $\alpha = if_1$. Thus N/L is 2-presented and hence by [14, Lemma 2.7], $\mathrm{Tor}_1(N/L, D^*) \cong (\mathrm{Ext}^1(N/L, D))^* = 0$. Therefore, $D^* \in$ SAS-N-$\mathbb{F}$ and consequently (SAS-N-$\mathbb{I}$)$^* \subseteq$SAS-N-$\mathbb{F}$. $\square$

Corollary 2.5. *The following conditions are equivalent for SAS-$\mathbb{F}$.*
(1) SAS-$\mathbb{F}$ is a definable class.
(2) SAS-$\mathbb{F}$ is c.u.d.p.
(3) $R^I \in$ SAS-$\mathbb{F}$, for each index I.
(4) All finitely generated semiartinian small left ideals of R are finitely presented.

(5) $(SAS\text{-}\mathbb{I})^* \subseteq SAS\text{-}\mathbb{F}$.
(6) $(SAS\text{-}\mathbb{I})^{**} \subseteq SAS\text{-}\mathbb{I}$.
(7) $(SAS\text{-}\mathbb{F})^{**} \subseteq SAS\text{-}\mathbb{F}$.

Proof. These follow by taking $N = {}_RR$ and applying Proposition 2.4. □

Theorem 2.6. *If a ring R is commutative, then the following conditions are equivalent for $SAS\text{-}\mathbb{F}$.*
(1) $SAS\text{-}\mathbb{F}$ is a definable class.
(2) $\mathrm{Hom}_R(D,V) \in SAS\text{-}\mathbb{F}$*, for every* $D \in SAS\text{-}\mathbb{I}$ *and* $V \in \mathbb{I}nj$.
(3) $\mathrm{Hom}_R(D,V) \in SAS\text{-}\mathbb{F}$*, for all* $D, V \in \mathbb{I}nj$.
(4) $\mathrm{Hom}_R(D,V) \in SAS\text{-}\mathbb{F}$*, for all* $D \in \mathbb{P}roj$ *and all* $V \in SAS\text{-}\mathbb{F}$.
(5) $\mathrm{Hom}_R(D,V) \in SAS\text{-}\mathbb{F}$*, for all* $D, V \in \mathbb{P}roj$.

Proof. (1)⇒(2). Let D be SAS-injective module and let V be injective module. Let $U \leq^{sas} R$ and let U be finitely generated. Since SAS-$\mathbb{F}$ is definable, U is finitely presented (by Corollary 2.5). Thus the sequence $0 \to \mathrm{Hom}_R(R/U, D) \to \mathrm{Hom}_R(R, D) \to \mathrm{Hom}_R(U, D) \to \mathrm{Ext}^2(R/U, D) = 0$ is exact. Since V is injective, the exact sequence $0 \to \mathrm{Hom}_R(D,V) \bigotimes_R U \to \mathrm{Hom}_R(D,V) \bigotimes_R R \to \mathrm{Hom}_R(D,V) \bigotimes_R (R/U) \to 0$ by [13, Theorem 3.2.11, p. 78]. So, $\mathrm{Hom}_R(D,V)$ is SAS-flat (by [10, Corollary 2.7]).
(2)⇒(3). Clear.
(3)⇒(1). For any index set S, we have from [1, Proposition 2.3.4, p. 66] and [14, Theorem 2.75, p. 92] that $(R^{**})^S \cong (\mathrm{Hom}_R(R^*, R^*))^S$. By [15, 11.10 (2), p. 87] and injectivity of R^* and $(R^*)^S$, we have $(R^{**})^S \cong \mathrm{Hom}_R(R^*, (R^*)^S) \in$ SAS-$\mathbb{F}$ for any index set S. By Corollary 2.2, $R^S \in$ SAS-$\mathbb{F}$ for any index set S. Thus (1) holds by Corollary 2.5.
(1)⇒(4). Let $D \in \mathbb{P}$roj and $V \in$ SAS-$\mathbb{F}$. Thus $D \oplus W \cong R^{(S)}$ for some a projective R-module W. So, $\mathrm{Hom}_R(D,V) \oplus \mathrm{Hom}_R(W,V) \cong \mathrm{Hom}_R(R^{(S)}, V) \cong (\mathrm{Hom}_R(R,V))^S \cong V^S$ by [15, 11.10 and 11.11, p. 87 and 88]. But V^S is SAS-flat by (1), thus $\mathrm{Hom}_R(D,V)$ is SAS-flat.
(4)⇒(5). Clear.
(5)⇒(1). By [15, 11.10 and 11.11, pp. 87, 88] and Corollary 2.5. □

Corollary 2.7. *For a commutative ring R, the next statement are equivalent if $SAS\text{-}\mathbb{I}$ is c.u.p.s. and $SAS\text{-}\mathbb{F}$ is a definable class.*
(1) $D \in SAS\text{-}\mathbb{I}$.
(2) $\mathrm{Hom}_R(D,W) \in SAS\text{-}\mathbb{F}$*, for each* $W \in \mathbb{I}nj$.
(3) $D \bigotimes_R W \in SAS\text{-}\mathbb{I}$*, for any* $W \in \mathbb{F}lat$.

Proof. (1)⇒(2). Use Theorem 2.6.
(2)⇒(3). By [15, Theorem 2.75, p. 92], $(D \otimes_R W)^* \cong \mathrm{Hom}_R(D, W^*)$ for

every R-module W. Let W be flat. Then $(D \otimes_R W)^*$ is SAS-flat by (2) and hence [10, Theorem 2.3] implies that $(D \otimes_R W)^{**}$ is SAS-injective. By hypothesis, $D \otimes_R W$ is SAS-injective.
(3)⇒(1). This follows from the hypothesis and [1, Proposition 2.3.4, p. 66]. □

Proposition 2.8. *The following conditions are equivalent when SAS-$\mathbb{F}$ is a definable class and SAS-$\mathbb{I}$ is c.u.p.s.*
(1) $R_R \in$ SAS-$\mathbb{I}$.
(2) If $W \in \mathbb{I}nj$, then $W \in$ SAS-$\mathbb{F}$.
(3) If $D \in \mathbb{F}lat$, then $D \in$ SAS-$\mathbb{I}$.

Proof. (1)⇒(2). Use [9, Proposition 4.2].
(2)⇒(3). This follows from the hypothesis and [10, Theorem 2.3].
(3)⇒(1). Since R_R is flat, the proof is obvious. □

Theorem 2.9. *For a commutative ring R, if SAS-$\mathbb{I}$ is c.u.p.s. and SAS-$\mathbb{F}$ is definable, then the following conditions are equivalent:*
(1) $R \in$ SAS-$\mathbb{I}$.
(2) If $M \in \mathbb{P}roj$ and $N \in \mathbb{F}lat$, then $\mathrm{Hom}_R(M, N) \in$ SAS-$\mathbb{I}$.
(3) If $M, N \in \mathbb{P}roj$, then $\mathrm{Hom}_R(M, N) \in$ SAS-$\mathbb{I}$.
(4) If $M, N \in \mathbb{I}nj$, then $\mathrm{Hom}_R(M, N) \in$ SAS-$\mathbb{I}$.

Proof. (1)⇒(2). This follows from Proposition 2.8 and [15, 11.10 and 11.11, pp. 87-88].
(2)⇒(3). Clear.
(3)⇒(1). By [15, 11.11, p. 88], $R \cong \mathrm{Hom}_R(R, R)$ and hence $R \in$ SAS-$\mathbb{I}$.
(1)⇒(4). This follows from [13, Theorem 3.2.1, p. 75], Proposition 2.8, and [11, Proposition 2.7].
(4)⇒(1). Apply [13, Theorem 3.2.1, p. 75], [1, Proposition 2.3.4, p. 66], [14, Theorem 2.75, p. 92], and Proposition 2.8. □

Acknowledgment. The authors would like to thank the referees for their valuable suggestions and comments.

References

[1] P. E. Bland, Rings and Their Modules, Walter de Gruyter and Co., Berlin, 2011.

[2] A. R. Mehdi, On a class of soc-injective modules, Iranian Journal of Mathematical Sciences and Informatics, **18,** no. 2, (2023), 51–65.

[3] H. H. Chyad, A. R. Mehdi, On SAS-injective rings, Iraqi Journal of Science, **65,** no. 7, (2024).

[4] A. R. Mehdi, Purity relative to classes of finitely presented modules, J. Algebra Appl., **12,** no. 8, (2013), 1350050, 1350050-26.

[5] A. R. Mehdi, On L-injective modules, Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki, **28,** no. 2, (2018), 176–192.

[6] A. R. Mehdi, D. T. Abd Al-Kadhim, (ρ, m)-N-injective modules, J. Phys.: Conf. Ser. **1234 012098**, (2019), 1–10.

[7] A. R. Mehdi, M. Prest, Almost dual pairs and definable classes of modules, Comm. Algebra **43,** no. 4, (2015), 1387–1397.

[8] E. A. Naim, A. R. Mehdi, On gs-pseudo-injective modules, Journal of Discrete Mathematical Sciences, **25,** no. 5, (2022), 1535–1545.

[9] Z. Zhanmin, I-n-Coherent rings, I-n-Semihereditary rings, and I-regular rings, Ukrainian Math. J., **66,** no. 6, (2014), 857–883.

[10] A. R. Obaid, A. R. Mehdi, SAS-N-flat modules, Journal of Discrete Mathematical Sciences and Cryptography, to appear.

[11] H. H. Chyad, A. R. Mehdi, SAS-injective modules, Journal of Al-Qadisiyah for Computer Science and Mathematics, **15,** no. 2, (2023), 31–40.

[12] A. R. Mehdi, Purity relative to classes of finitely presented modules, PhD Thesis, Manchester University, (2013).

[13] E. E. Enochs, O. M. G. Jenda, Relative Homological Algebra, Walter de Gruyter, 2000.

[14] J. J. Rotman, An Introduction to Homological Algebra, Springer, 2009.

[15] R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach, 1991.