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# Jacobi polynomials and bi-univalent functions

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**Abstract**

In this paper, we present a novel qualitative class of analytic and bi-univalent functions associated with Jacobi polynomials. Moreover, we establish bounds on coefficients for functions within this class and address the Fekete-Szegö problem. Furthermore, exploring the parameters in our primary findings yields a diverse array of new results.

## 1 Introduction and preliminaries

In 1784, Legendre [1] introduced orthogonal polynomials which are frequently used in solving ordinary differential equations with specific model constraints and which play a crucial role in approximation theory [2].

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Two polynomials $Y_n$ and $Y_m$ of order $n$ and $m$, respectively, are said to be orthogonal if

$$\int_{\epsilon}^{\iota} Y_n(x)Y_m(x)v(x)dx = 0, \qquad \text{for} \qquad n \neq m, \tag{1.1}$$

Assuming $v(x)$ is non-negative in the interval $(\epsilon, \iota)$, all polynomials of finite order $Y_n(x)$ possess a clearly defined integral. Jacobi polynomials belong to the category of orthogonal polynomials.

In this paper, we analytically examine a newly defined subclass $\mathbf{B}_\Sigma(\alpha, \beta, x)$ of bi-univalent functions utilizing Jacobi polynomials.

Let $\mathcal{A}$ represent the class of analytic functions $\Theta$ in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, normalized with $\Theta(0) = 0$ and $\Theta'(0) = 1$. Consequently, each $\Theta \in \mathcal{A}$ can be expressed as:

$$\Theta(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \quad (z \in \mathbb{U}). \tag{1.2}$$

Moreover, the set of all univalent functions $\Theta \in \mathcal{A}$ is denoted by $\mathcal{S}$ (for more information, the reader is referred to [3]).

The field of geometric function theory stands to gain significant advantages from the robust tools offered by differential subordination of analytic functions. The initial differential subordination problem was introduced by Miller and Mocanu [4], with further references provided in [5]. The comprehensive developments in this area have been documented in Miller and Mocanu's book [6], including publication dates.

It is well established that for an analytic and univalent function $\Theta(z)$ mapping a domain $\mathbb{D}1$ onto a domain $\mathbb{D}2$, the inverse function $g(z) = \Theta^{-1}(z)$ is defined as

$$g\left(\Theta(z)\right) = z, \quad (z \in \mathbb{D}_1),$$

is analytic and univalent. Moreover, (see [3]), every function $\Theta \in \mathcal{S}$ has an inverse map $\Theta^{-1}$ satisfying

$$\Theta^{-1}(\Theta(z)) = z \quad (z \in \mathbb{U}),$$

and

$$\Theta\left(\Theta^{-1}(\varpi)\right) = \varpi \quad \left(|\varpi| < r_0(\Theta); r_0(\Theta) \geq \frac{1}{4}\right).$$

In fact, the inverse function is given by

$$\Theta^{-1}(\varpi) = \varpi - a_2^2\varpi + (2a_2^2 - a_3)\varpi^3 - (5a_2^3 - 5a_2a_3 + a_4)\varpi^4 + \cdots. \tag{1.3}$$

A function $\Theta \in \mathcal{A}$ is bi-univalent in $\mathbb{U}$ if both $\Theta(z)$ and $\Theta^{-1}(z)$ are univalent in $\mathbb{U}$. Let $\Sigma$ represent the class of bi-univalent functions in $\mathbb{U}$ as defined in (1.2). For further details on the class $\Sigma$, the interested reader is referred to [7, 8, 9, 10, 11, 12, 13, 14, 15].

For nonnegative $n$, $n+\vartheta$, $n+\varsigma$, a generating function of Jacobi polynomials is defined by

$$J_n(x,z) = 2^{\vartheta+\varsigma} R^{-1} \left(1-x+R\right)^{-\vartheta} \left(1+x+R\right)^{-\varsigma},$$

where $R = R(x,z) = (1-2zx+x^2)^{0.5}$, $\vartheta > -1$, $\varsigma > -1$, $x \in [-1,1]$ and $z \in \mathbb{U}$, (see [16]).

For a fixed $x$, the function $J_n(x,z)$ is analytic in $\mathbb{U}$ and so is represented by a Taylor series expansion as follows:

$$J_n(x,z) = \sum_{n=0}^{\infty} P_n^{(\vartheta,\varsigma)}(x) z^n, \tag{1.4}$$

where $P_n^{(\vartheta,\varsigma)}(x)$ is Jacobi polynomial of degree $n$.

The Jacobi polynomial $P_n^{(\vartheta,\varsigma)}(x)$ satisfies a second-order linear homogeneous differential equation:

$$(1-x^2)y'' + (\varsigma - \vartheta - (\vartheta+\varsigma+2)x)y' + n(n+\vartheta+\varsigma+1)y = 0.$$

Jacobi polynomials can alternatively be characterized by the following recursive relationships:

$$P_n^{(\vartheta,\varsigma)}(x) = (a_{n-1}z - b_{n-1})P_{n-1}^{(\vartheta,\varsigma)}(x) - c_{n-1}P_{n-2}^{(\vartheta,\varsigma)}(x),\ n \geqslant 2,$$

where $a_n = \frac{(2n+\vartheta+\varsigma+1)(2n+\vartheta+\varsigma+2)}{2(n+1)(n+\vartheta+\varsigma+1)}$, $b_n = \frac{(2n+\vartheta+\varsigma+1)\left(\varsigma^2-\vartheta^2\right)}{2(n+1)(n+\vartheta+\varsigma+1)(2n+\vartheta+\varsigma)}$ and $c_n = \frac{(2n+\vartheta+\varsigma+2)(n+\vartheta)(n+\varsigma)}{(n+1)(n+\vartheta+\varsigma+1)(2n+\vartheta+\varsigma)}$,
with the initial values

$$P_0^{(\vartheta,\varsigma)}(x) = 1,\ P_1^{(\vartheta,\varsigma)}(x) = (\vartheta+1) + \frac{1}{2}(\vartheta+\varsigma+2)(x-1) \text{ and} \tag{1.5}$$
$$P_2^{(\vartheta,\varsigma)}(x) = \frac{(\vartheta+1)(\vartheta+2)}{2} + \frac{1}{2}(\vartheta+2)(\vartheta+\varsigma+3)(x-1) + \frac{1}{8}(\vartheta+\varsigma+3)(\vartheta+\varsigma+4)(x-1)^2.$$

To begin, we introduce certain special instances of the polynomials $P_n^{(\vartheta,\varsigma)}$:

1. For $\vartheta = \varsigma = 0$, we get the Legendre Polynomials.

2. For $\vartheta = \varsigma = -0.5$, this results in the Chebyshev Polynomials of the first kind.

3. For $\vartheta = \varsigma = 0.5$, this results in the Chebyshev Polynomials of the second kind.

4. For $\vartheta = \varsigma$, we get the Gegenbauer Polynomials and each is replaced by $(\vartheta - 0.5)$.

Ezrohi [17] introduced the class $\mathcal{U}(\varepsilon)$ as follows:

$$\mathcal{U}(\varepsilon) = \{\Theta : \ \Theta \in \mathcal{S} \ \text{ and } \ \mathrm{Re}\{\Theta'(z)\} > \varepsilon, \quad (z \in \mathbb{U}; 0 \leq \varepsilon < 1)\}.$$

In recent times, numerous scholars have delved into the realm of bi-univalent functions linked to orthogonal polynomials. Notable references in this area include [18, 19, 20, 21, 22, 23, 24, 26, 27, 28, 29]. However, when it comes to Jacobi polynomials, to the best of our knowledge, there has been a dearth of research on bi-univalent functions in the existing literature. The primary objective of this paper is to kickstart an investigation into the characteristics of bi-univalent functions associated with Jacobi polynomials. To achieve this objective, we will consider the following definitions.

# 2 Definition

Definition 2.1 characterizes a class of bi-univalent functions, which are closed to convex, in terms of Jacobi polynomials.

**Definition 2.1.** *Let $\vartheta > -1$, $\varsigma > -1$, $x \in (\frac{1}{2}, 1]$ and $n$, $n + \vartheta$, $n + \varsigma$ be nonnegative integers. A function $\Theta \in \Sigma$ given by (1.2) is said to be in the class $\mathbf{B}_\Sigma(\vartheta, \varsigma, x)$ if the following subordinations are satisfied:*

$$\Theta'(z) \prec J_n(x, z) \tag{2.1}$$

*and*

$$g'(\varpi) \prec J_n(x, \varpi), \tag{2.2}$$

*where the function $J_n$ is given by (1.4) and the function $g(\varpi) = \Theta^{-1}(\varpi)$ is defined by (1.3).*

Unless otherwise mentioned, we assume that $\vartheta > -1$, $\varsigma > -1$, $x \in (\frac{1}{2}, 1]$ and $n$, $n + \vartheta$, $n + \varsigma$ are nonnegative integers.

# 3 Coefficient bounds of the subclass $\mathbf{B}_\Sigma(\vartheta, \varsigma, x)$

In this section, our focus is on determining initial coefficient bounds for the subclass $\mathbf{B}_\Sigma(\vartheta, \varsigma, x)$.

**Theorem 3.1.** *Let $f \in \Sigma$ given by (1.2) belong to the class $\mathbf{B}_\Sigma(\vartheta, \varsigma, x)$. Then*

$$|a_2| \leq \frac{\left|(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right| \sqrt{\left|(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right|}}{\sqrt{3-2\Upsilon(x,\vartheta,\varsigma)}},$$

*and*

$$|a_3| \leq \frac{\left[(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right]^2}{4} + \frac{(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)}{3},$$

*where*

$$\Upsilon(x,\vartheta,\varsigma) = (\vartheta+1)(\vartheta+2)+(\vartheta+2)(\vartheta+\varsigma+3)(x-1) + \frac{1}{4}(\vartheta+\varsigma+3)(\vartheta+\varsigma+4)(x-1)^2.$$

*Proof.* Suppose $\Theta \in \mathbf{B}_\Sigma(\vartheta, \varsigma, x)$. According to equations (2.1) and (2.2), for all $z, \varpi \in \mathbb{U}$ and analytic functions $r$ and $s$ with $r(0) = s(0) = 0$ and $|r(z)| < 1$, $|s(\varpi)| < 1$, we have:

$$\Theta'(z) = J_n(x, r(z)) \tag{3.1}$$

and

$$g'(\varpi) = J_n(x, s(\varpi)). \tag{3.2}$$

Thus

$$\Theta'(z) = 1 + P_1^{(\vartheta,\varsigma)}(x) b_1 z + \left[P_1^{(\vartheta,\varsigma)}(x) b_2 + P_2^{(\vartheta,\varsigma)}(x) b_1^2\right] z^2 + \cdots \tag{3.3}$$

and

$$g'(\varpi) = 1 + P_1^{(\vartheta,\varsigma)}(x) d_1 \varpi + \left[P_1^{(\vartheta,\varsigma)}(x) d_2 + P_2^{(\vartheta,\varsigma)}(x) d_1^2\right])\varpi^2 + \cdots. \tag{3.4}$$

It is well known that if

$$|r(z)| = \left|b_1 z + b_2 z^2 + b_3 z^3 + \cdots\right| < 1, \quad (z \in \mathbb{U})$$

and

$$|s(\varpi)| = \left|d_1\varpi + d_2^2\varpi + d_3^3\varpi + \cdots\right| < 1, \quad (\varpi \in \mathbb{U}),$$

then

$$|b_j| \leq 1 \text{ and } |d_j| \leq 1 \text{ for all } j \in \mathbb{N}. \tag{3.5}$$

Comparing the coefficients in (3.3) and (3.4), we get

$$2a_2 = P_1^{(\vartheta,\varsigma)}(x)b_1, \tag{3.6}$$

$$3a_3 = P_1^{(\vartheta,\varsigma)}(x)b_2 + P_2^{(\vartheta,\varsigma)}(x)b_1^2, \tag{3.7}$$

$$-2a_2 = P_1^{(\vartheta,\varsigma)}(x)d_1, \tag{3.8}$$

and

$$3\left[2a_2^2 - a_3\right] = P_1^{(\vartheta,\varsigma)}(x)d_2 + P_2^{(\vartheta,\varsigma)}(x)d_1^2. \tag{3.9}$$

From (3.6) and (3.8), it follows that

$$b_1 = -d_1 \tag{3.10}$$

and

$$8a_2^2 = \left[P_1^{(\vartheta,\varsigma)}(x)\right]^2 \left(b_1^2 + d_1^2\right). \tag{3.11}$$

If we add (3.7) and (3.9), we get

$$6a_2^2 = P_1^{(\vartheta,\varsigma)}(x)\left(b_2 + d_2\right) + P_2^{(\vartheta,\varsigma)}(x)\left(b_1^2 + d_1^2\right). \tag{3.12}$$

Substituting the value of $(b_1^2 + d_1^2)$ from (3.11) into the right hand side of (3.12), we get

$$2\left[3 - \frac{4P_2^{(\vartheta,\varsigma)}(x)}{\left[P_1^{(\vartheta,\varsigma)}(x)\right]^2}\right] a_2^2 = P_1^{(\vartheta,\varsigma)}(x)\left(b_2 + d_2\right). \tag{3.13}$$

Using (1.5), (3.5) and (3.13), we find that

$$|a_2| \leq \frac{\left|(\vartheta+1) + \frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right| \sqrt{\left|(\vartheta+1) + \frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right|}}{\sqrt{3 - 2\Upsilon(x,\vartheta,\varsigma)}},$$

where

$$\Upsilon(x,\vartheta,\varsigma) = (\vartheta+1)(\vartheta+2) + (\vartheta+2)(\vartheta+\varsigma+3)(x-1) + \frac{1}{4}(\vartheta+\varsigma+3)(\vartheta+\varsigma+4)(x-1)^2.$$

Moreover, if we subtract (3.9) from (3.7), we get

$$6\left(a_3 - a_2^2\right) = P_1^{(\vartheta,\varsigma)}(x)\left(b_2 - d_2\right) + P_2^{(\vartheta,\varsigma)}(x)\left(b_1^2 - d_1^2\right). \tag{3.14}$$

In view of (3.11) and (3.14),

$$a_3 = \frac{\left[P_1^{(\vartheta,\varsigma)}(x)\right]^2}{8}\left(b_1^2 + d_1^2\right) + \frac{P_1^{(\vartheta,\varsigma)}(x)}{6}\left(b_2 - d_2\right).$$

Applying (1.5) and (3.5), we have

$$|a_3| \leq \frac{\left[(\vartheta+1) + \frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right]^2}{4} + \frac{(\vartheta+1) + \frac{1}{2}(\vartheta+\varsigma+2)(x-1)}{3}.$$

Thee proof is complete. □

# 4 Fekete–Szegö problem for the subclass $\mathbf{B}_\Sigma(\vartheta, \varsigma, x)$

A prominent issue concerning coefficients of univalent analytic functions is the Fekete-Szegő inequality. Initially proposed by [25], it asserts that for $\Theta \in \Sigma$, where $\Sigma$ denotes

$$|a_3 - \tau a_2^2| \leq 1 + 2e^{-2\tau/(1-\mu)}.$$

When $\tau$ is real, this bound is sharp.

In this section, for functions in the class $\mathbf{B}_\Sigma(\vartheta, \varsigma, x)$ we provide Fekete–Szegö inequalities.

**Theorem 4.1.** *Let $\Theta \in \Sigma$ given by (1.2) belong to the class $\mathbf{B}_\Sigma(\vartheta, \varsigma, x)$. Then*

$$\left|a_3 - \tau a_2^2\right| \leq \begin{cases} \dfrac{\left|(\vartheta+1) + \frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right|}{3}, & |\tau - 1| \leq k(x) \\[2ex] \dfrac{|1-\tau|\left|(\vartheta+1) + \frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right|^3}{\left[\begin{array}{c} 3\left[(\vartheta+1) + \frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right]^2 \\ -2\left(\begin{array}{c}(\vartheta+1)(\vartheta+2) + (\vartheta+2)(\vartheta+\varsigma+3)(x-1) \\ +\frac{1}{4}(\vartheta+\varsigma+3)(\vartheta+\varsigma+4)(x-1)^2\end{array}\right)\end{array}\right]}, & |\tau - 1| \geq k(x) \end{cases}$$

*where*

$$k(x)=\left|1-\frac{4\left[\frac{(\vartheta+1)(\vartheta+2)}{2}+\frac{1}{2}(\vartheta+2)(\vartheta+\varsigma+3)(x-1)+\frac{1}{8}(\vartheta+\varsigma+3)(\vartheta+\varsigma+4)(x-1)^{2}\right]}{3\left[(\vartheta+1)+\frac{1}{2}(\vartheta+\varsigma+2)(x-1)\right]^{2}}\right|.$$

*Proof.* From (3.13) and (3.14)

$$\begin{aligned} a_3-\tau a_2^2 &= \frac{(1-\tau)\left[P_1^{(\vartheta,\varsigma)}(x)\right]^3(b_2+d_2)}{2\left[3\left[P_1^{(\vartheta,\varsigma)}(x)\right]^2-4P_2^{(\vartheta,\varsigma)}(x)\right]}+\frac{P_1^{(\vartheta,\varsigma)}(x)}{6}(b_2-d_2)\\ &= P_1^{(\vartheta,\varsigma)}(x)\left[\left(h(\tau)+\frac{1}{6}\right)b_2+\left(h(\tau)-\frac{1}{6}\right)d_2\right], \end{aligned}$$

where

$$h(\tau)=\frac{(1-\tau)\left[P_1^{(\vartheta,\varsigma)}(x)\right]^2}{2\left[3\left[P_1^{(\vartheta,\varsigma)}(x)\right]^2-4P_2^{(\vartheta,\varsigma)}(x)\right]},$$

In view of (1.5), we have

$$\left|a_3-\tau a_2^2\right|\le\begin{cases}\frac{\left|P_1^{(\vartheta,\varsigma)}(x)\right|}{3} & 0\le|h(\tau)|\le\frac{1}{6},\\ 2\left|P_1^{(\vartheta,\varsigma)}(x)\right||h(\tau)| & |h(\tau)|\ge\frac{1}{6}.\end{cases}$$

The proof is now complete. □

# 5 Corollaries and Consequences

In this section, we use our main results to derive each of the new corollaries and implications that follow.

**Corollary 5.1.** *Let $\Theta\in\Sigma$ given by (1.2) belong to the class $\mathbf{B}_\Sigma(\vartheta,\vartheta,x)=\mathbf{B}_\Sigma(\vartheta,x)$. Then*

$$|a_2|\le\frac{|(\vartheta+1)+(\vartheta+1)(x-1)|\sqrt{|(\vartheta+1)+(\vartheta+1)(x-1)|}}{\sqrt{3-2\Upsilon(x,\vartheta,\varsigma)}},$$

$$|a_3| \leq \frac{[(\vartheta+1)+(\vartheta+1)(x-1)]^2}{4} + \frac{(\vartheta+1)+(\vartheta+1)(x-1)}{3},$$

*and*

$$\left|a_3 - \tau a_2^2\right| \leq \begin{cases} \frac{|(\vartheta+1)+(\vartheta+1)(x-1)|}{3}, & |\tau-1| \leq \Theta(x) \\ \frac{|1-\tau||(\vartheta+1)+(\vartheta+1)(x-1)|^3}{\left[\begin{array}{c} 3\left[(\vartheta+1)+(\vartheta+1)(x-1)\right]^2 \\ -2\left(\begin{array}{c}(\vartheta+1)(\vartheta+2)+(\vartheta+2)(2\vartheta+3)(x-1) \\ +\frac{1}{2}(2\vartheta+3)(\vartheta+2)(x-1)^2\end{array}\right)\end{array}\right]}, & |\tau-1| \geq \Theta(x) \end{cases}$$

*where*

$$\Upsilon(x,\vartheta,\varsigma) = (\vartheta+1)(\vartheta+2) + (\vartheta+2)(2\vartheta+3)(x-1) + \frac{1}{2}(2\vartheta+3)(\vartheta+2)(x-1)^2,$$

*and*

$$\Theta(x) = \left|1 - \frac{4\left[\frac{(\vartheta+1)(\vartheta+2)}{2} + \frac{1}{2}(\vartheta+2)(2\vartheta+3)(x-1) + \frac{1}{4}(2\vartheta+3)(\vartheta+2)(x-1)^2\right]}{3\left[(\vartheta+1)+(\vartheta+1)(x-1)\right]^2}\right|$$

**Corollary 5.2.** *Let $\Theta \in \Sigma$ given by (1.2) belong to the class $\mathbf{B}_\Sigma(0,0,x) = \mathbf{B}_\Sigma(x)$. Then*

$$|a_2| \leq \frac{|1+(x-1)|\sqrt{|1+(x-1)|}}{\sqrt{3-2(2+6(x-1)+3(x-1)^2)}},$$

$$|a_3| \leq \frac{[1+(x-1)]^2}{4} + \frac{1+(x-1)}{3},$$

*and*

$$\left|a_3 - \tau a_2^2\right| \leq \begin{cases} \frac{|1+(x-1)|}{3}, & |\tau-1| \leq Q(x) \\ \frac{|1-\tau||1+(x-1)|^3}{\left[\begin{array}{c} 3\left[1+(x-1)\right]^2 \\ -2\left(\begin{array}{c}2+6(x-1) \\ +3(x-1)^2\end{array}\right)\end{array}\right]}, & |\tau-1| \geq Q(x) \end{cases}$$

*where*

$$Q(x) = \left|1 - \frac{4\left[1+3(x-1)+\frac{3}{2}(x-1)^2\right]}{3\left[1+(x-1)\right]^2}\right|.$$

# 6 Concluding Remark

In this study, we presented and explored the coefficient-related concerns associated with the newly introduced subclasses $\mathbf{B}_{\Sigma}(\vartheta, \varsigma, x)$, $\mathbf{B}_{\Sigma}(\vartheta, x)$, and $\mathbf{B}_{\Sigma}(x)$ within the bi-univalent functions class defined in the open unit disk $\mathbb{U}$. Our investigation involved estimating the Fekete-Szegő functional problems and the Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions within each of these bi-univalent function subclasses. Moreover, the specialization of parameters in our main results yielded numerous novel findings.

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