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The Commutativity of Quotient Rings with Homoderivations

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Abstract

The main goal of this paper is to propose several algebraic identities concerning homoderivations with a prime ideal to establish the commutativity of the quotient ring $\mathfrak{S}/\mathfrak{P}$. Moreover, we generalize the results of El-Sofy [10] and Melaibari et al. [11] by proposing that the identity: $[\mathcal{G}(\breve{\imath}), \breve{\imath}]$ which belongs to a prime ideal $\mathfrak{P}$ for every element $\breve{\imath}$ of a nonzero ideal $\mathfrak{L}$. Furthermore, we use two homoderivations to prove the same commutativity.

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1 Introduction

Throughout this article, $\mathfrak{P}$ is a prime ideal of an associative ring $\mathfrak{S}$, with $\mathfrak{P}$ being a proper subset of the nonzero ideal $\mathfrak{L}$ of $\mathfrak{S}$, denoted by $\mathfrak{P} \subsetneq \mathfrak{L}$. A proper ideal $\mathfrak{P}$ of $\mathfrak{S}$ is prime if the product $\check{\imath}_1\check{\imath}_2$ is contained in $\mathfrak{P}$ for all pairs of elements $\check{\imath}_1, \check{\imath}_2$ belonging to $\mathfrak{S}$, then at least one of $\check{\imath}_1$ or $\check{\imath}_2$ is also an element of $\mathfrak{P}$. Moreover, if the zero ideal (0) of a ring $\mathfrak{S}$ is considered as prime, then $\mathfrak{S}$ is termed a prime ring. In $\mathfrak{S}$, the expression $[\check{\imath}_1, \check{\imath}_2]$ represents the commutator $\check{\imath}_1\check{\imath}_2 - \check{\imath}_2\check{\imath}_1$ for elements $\check{\imath}_1$ and $\check{\imath}_2$. Posner's definition of a derivation, introduced in 1957, ignited the interest of numerous authors who delved into derivation theory across various algebraic structures, all based on Posner's original definition of derivation on a ring [1]. The significance of derivatives in matrix eigenvalue computations, quantum physics, business, and engineering calculations has led to the emergence of various types and generalizations of derivations [2, 3, 4, 5]. In the theory of commutative rings, the concept of prime ideals is of significant importance. Several authors delved into the commutativity of the quotient ring $\mathfrak{S}/\mathfrak{P}$ by studying the action of various types of derivations fulfilling different algebraic identities involving the prime ideal $\mathfrak{P}$ [6, 7, 8, 9]. In 2000, El-Sofy [10] defined a homoderivation (HD, for short) on $\mathfrak{S}$ as an additive mapping $\mathcal{G}$ that satisfies $\mathcal{G}(\check{\imath}_1\check{\imath}_2) = \mathcal{G}(\check{\imath}_1)\mathcal{G}(\check{\imath}_2)+\mathcal{G}(\check{\imath}_1)\check{\imath}_2+\check{\imath}_1\mathcal{G}(\check{\imath}_2)$ for all elements $\check{\imath}_1, \check{\imath}_2$ belonging to $\mathfrak{S}$. The term of HD mapping is the outcome of merging the ideas of homomorphisms and derivations. An HD mapping $\mathcal{G}$ is a derivation if $\mathcal{G}(\check{\imath}_1)\mathcal{G}(\check{\imath}_2) = 0$ for all $\check{\imath}_1, \check{\imath}_2$ in $\mathfrak{S}$. Moreover, the zero mapping in a prime ring $\mathfrak{S}$ is the only additive mapping that serves as both HD and a derivation mapping. A mapping $\mathfrak{z}$ of $\mathfrak{S}$ to itself is zero-power valued (Z-PV, for short) on a subset $\mathfrak{A}$ of $\mathfrak{S}$ if $\mathfrak{z}(\mathfrak{A})$ is contained in $\mathfrak{A}$ and there exists a positive integer $\aleph(\wp) > 1$ such that $\mathfrak{z}^{\aleph(\wp)}(\wp) = 0$, also the commutativity of specific ring types equipped with HD [10]. Melaibari et al. [11] studied the commutativity of rings admitting an HD mapping $\mathcal{G}$ such that $\mathcal{G}([\check{\imath}_1, \check{\imath}_2]) = 0$ for all the elements $\imath_1$ and $\imath_2$ belonging to a suitable subset of $\mathfrak{S}$. Several mathematicians discussed the commutator of HD mappings, when it is identical to zero, to prove the commutativity property [12, 13].

Algebra reach extends across multiple facets of life, wielding significant influence in numerous scientific disciplines and maintaining essential relevance throughout all mathematical domains. Disciplines like abstract and applied algebra are closely interwoven with diverse scientific fields where computer science and engineering science stand out as pivotal players [15, 16, 17, 18, 14].

We examine the behavior of HD mappings by introducing numerous algebraic identities. These identities encompass various HD mappings that are included in prime ideals thereby extending the findings of El-Sofy and Melaibari.

2 Main Results

In the following theorem, we discuss the behavior of HD mapping for the identity $[\mathcal{G}(\check{\imath}), \check{\imath}]$ involving the prime ideal $\mathfrak{P}$:

Theorem 2.1. *Let $\mathcal{G}$ be an HD mapping of $\mathfrak{S}$, and a Z-PV mapping on $\mathfrak{L}$. For every element $\check{\imath}$ of $\mathfrak{L}$, if the commutator $[\mathcal{G}(\check{\imath}), \check{\imath}]$ belongs to $\mathfrak{P}$, then either $\mathfrak{S}/\mathfrak{P}$ is commutative or $\mathcal{G}(\mathfrak{L})$ is contained in $\mathfrak{P}$.*

Proof.
By assumption, for every $\check{\imath} \in \mathfrak{L}$, we have

$$[\mathcal{G}(\check{\imath}), \check{\imath}] \in \mathfrak{P}. \tag{2.1}$$

Linearizing Equation (2.1) and using it yields

$$[\mathcal{G}(\check{\imath}_1), \check{\imath}_2] + [\mathcal{G}(\check{\imath}_2), \check{\imath}_1] \in \mathfrak{P} \text{ for all } \check{\imath}_1, \check{\imath}_2 \in \mathfrak{L}. \tag{2.2}$$

In Equation(2.2), substitute $\check{\imath}_1$ with $\check{\imath}_1\check{\imath}_2$ and use it with Equation (2.1) to get $[\check{\imath}_1 + \mathcal{G}(\check{\imath}_1), \check{\imath}_2]\mathcal{G}(\check{\imath}_2) \in \mathfrak{P}$, for all $\check{\imath}_1, \check{\imath}_2 \in \mathfrak{L}$. Replace $\check{\imath}_1$ in this equation with $\sum_{i=1}^{n}(-1)^{i-1}\mathcal{G}^{i-1}(\check{\imath}_1)$, where $\mathcal{G}^0(\mathfrak{S})$ is the identity mapping $id_{\mathfrak{S}}$. Since $\mathcal{G}$ is a Z-PV mapping on $\mathfrak{L}$, there exists an integer $\aleph(\check{\imath})$ greater than 1 for which $\mathcal{G}^{\aleph(\check{\imath})}(\check{\imath}) = 0$ for every $\check{\imath} \in \mathfrak{P}$. Therefore, $[\check{\imath}_1, \check{\imath}_2]\mathcal{G}(\check{\imath}_2) \in \mathfrak{P}$, for all $\check{\imath}_1, \check{\imath}_2 \in \mathfrak{L}$. When $\check{\imath}_1$ is substituted with $\check{\imath}_1\mathfrak{z}$, for $\mathfrak{z} \in \mathfrak{L}$ in this equation, we get $[\check{\imath}_1, \check{\imath}_2]\mathfrak{L}\mathcal{G}(\check{\imath}_2) \subseteq \mathfrak{P}$. By applying [7, Fact 1] to the last equation, either $[\check{\imath}_1, \check{\imath}_2] \in \mathfrak{P}$ or $\mathcal{G}(\check{\imath}_2) \in \mathfrak{P}$, for all $\check{\imath}_1, \check{\imath}_2 \in \mathfrak{L}$.
Let $\mathcal{X} = \{\omega \in \mathfrak{L} : [\omega, \mathfrak{L}] \subseteq \mathfrak{P}\}$ and $\mathcal{Y} = \{\omega \in \mathfrak{L} : \mathcal{G}(\omega) \subseteq \mathfrak{P}\}$. Undoubtedly, $\mathfrak{L} = \mathcal{X} \cup \mathcal{Y}$. Employing Brauer's trick fulfills either $\mathfrak{L} = \mathcal{X}$ or $\mathfrak{L} = \mathcal{Y}$. In light of [7, Remark], the first case leads to $\mathfrak{S}/\mathfrak{P}$ is commutative, whereas the second establishes $\mathcal{G}(\mathfrak{L}) \subseteq \mathfrak{P}$. □

The following corollary follows by considering $\mathfrak{P} = (0)$.

Corollary 2.2. *[10, Theoerm 3.4.7] Let $\mathcal{G}$ be a nonzero HD mapping on a prime ring $\mathfrak{S}$ that commutes and is Z-PV on $\mathfrak{L}$. Then $\mathfrak{S}$ is commutative.*

In the next corollary, two HD mappings are utilized to establish the commutativity of $\mathfrak{S}/\mathfrak{P}$.

Corollary 2.3. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be HD mappings of $\mathfrak{S}$. If $\mathcal{G}_2$ is Z-PV on $\mathfrak{L}$ that satisfies for all $\check{\imath}_1, \check{\imath}_2 \in \mathfrak{L}$, $\mathcal{G}_1[\check{\imath}_1, \check{\imath}_2] \pm [\mathcal{G}_2(\check{\imath}_1), \check{\imath}_2]$ belongs to $\mathfrak{P}$. Then either $\mathfrak{S}/\mathfrak{P}$ is commutative or $\mathcal{G}_1(\mathfrak{L})$ is contained in $\mathfrak{P}$.*

Proof.
By hypothesis,

$$\mathcal{G}_1[\check{\imath}_1, \check{\imath}_2] \pm [\mathcal{G}_2(\check{\imath}_1), \check{\imath}_2] \in \mathfrak{P} \text{ for all } \check{\imath}_1, \check{\imath}_2 \in \mathfrak{L}. \tag{2.3}$$

If $\check{\imath}_1 = \check{\imath}_2$ in Equation (2.3), the equation can be simplified to $[\mathcal{G}_2(\check{\imath}_1), \check{\imath}_1] \subseteq \mathfrak{P}$ for all $\check{\imath}_1 \in \mathfrak{L}$. The result now follows by Theorem 2.1. □

In Corollary 2.3, if $\mathfrak{P} = (0)$ and $\mathcal{G}_2 = 0$, then one can obtain the following corollary.

Corollary 2.4. *[11, Theoerm 3.2] For a nonzero HD mapping $\mathcal{G}$ of a prime ring $\mathfrak{S}$, if the condition $\mathcal{G}[\check{\imath}_1, \check{\imath}_2] = 0$ holds for all $\check{\imath}_1, \check{\imath}_2 \in \mathfrak{L}$, then $\mathfrak{S}$ is commutative.*

The following Lemmas are necessary to prove Theorem 2.7. Throughout, the symbol $\mathcal{G}_2^2(\mathfrak{a})$ represents the composition of $\mathcal{G}_2$ with itself (that is, $\mathcal{G}_2^2(\mathfrak{a}) = \mathcal{G}_2(\mathcal{G}_2(\mathfrak{a}))$).

Lemma 2.5. *Suppose that Char$(\mathfrak{S}/\mathfrak{P}) \neq 2$ and $\mathcal{G}$ is an HD mapping of $\mathfrak{S}$ such that $\mathcal{G}^2(\mathfrak{L}) \subseteq \mathfrak{P}$. Then $\mathcal{G}(\mathfrak{L}) \subseteq \mathfrak{P}$.*

Proof.
The result follows directly from the definition of HD and the primeness property of $\mathfrak{P}$ with Char$(\mathfrak{S}/\mathfrak{P})$ not equal to 2. □

Lemma 2.6. *Suppose that Char$(\mathfrak{S}/\mathfrak{P}) \neq 2$ and the mappings $\mathcal{G}_1$ and $\mathcal{G}_2$ are HD of $\mathfrak{S}$. If the composition $\mathcal{G}_1\mathcal{G}_2(\mathfrak{L})$ is contained in $\mathfrak{P}$, then either $\mathcal{G}_1(\mathfrak{L}) \subseteq \mathfrak{P}$ or $\mathcal{G}_2(\mathfrak{L}) \subseteq \mathfrak{P}$.*

Proof.
Since $\mathcal{G}_1\mathcal{G}_2(\mathfrak{L}) \subseteq \mathfrak{P}$ and $\mathcal{G}_2(\mathfrak{L}) \subseteq \mathfrak{L}$, $\mathcal{G}_1\mathcal{G}_2^2(\mathfrak{L}) \subseteq \mathfrak{P}$, where $\mathcal{G}_2^2(\mathfrak{L}) = \mathcal{G}_2(\mathcal{G}_2(\mathfrak{L}))$. So, for all elements $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathfrak{L}$, the following statement holds true

$$\mathcal{G}_1\mathcal{G}_2(\mathcal{G}_2(\mathfrak{a}_1)\mathfrak{a}_2) = \mathcal{G}_2^2(\mathfrak{a}_1)\mathcal{G}_1(\mathfrak{a}_2) \in \mathfrak{P}. \tag{2.4}$$

Substitute $\mathfrak{a}_2\mathfrak{a}_3$ for $\mathfrak{a}_2$ where $\mathfrak{a}_3 \in \mathfrak{L}$ in Equation 2.4 and use it, we obtain $\mathcal{G}_2^2(\mathfrak{a}_1)\mathfrak{L}\mathcal{G}_1(\mathfrak{a}_3) \in \mathfrak{P}$ for all $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3 \in \mathfrak{L}$. According to [7, Fact 1], the last equation leads to either $\mathcal{G}_2^2(\mathfrak{L}) \subseteq \mathfrak{P}$ or $\mathcal{G}_1(\mathfrak{L}) \in \mathfrak{P}$. Therefore, by Lemma 2.5, the first case implies that $\mathcal{G}_2(\mathfrak{L}) \subseteq \mathfrak{P}$, while the second case yields $\mathcal{G}_1(\mathfrak{L}) \subseteq \mathfrak{P}$. □

To investigate the commutativity of $\mathfrak{S}/\mathfrak{P}$, two HD mappings are considered in the following theorem:

Theorem 2.7. *Suppose that Char*$(\mathfrak{S}/\mathfrak{P}) \neq 2$ *and the mappings* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *are HD of* $\mathfrak{S}$. *If* $\mathcal{G}_1$ *is Z-PV on* $\mathfrak{L}$ *that satisfies for all* $\check{\imath}_1, \check{\imath}_2 \in \mathfrak{L}$, $\mathcal{G}_1[\check{\imath}_1, \check{\imath}_2] - [\mathcal{G}_2(\check{\imath}_1), \check{\imath}_2] - [\check{\imath}_1, \mathcal{G}_1(\check{\imath}_2)]$ *belongs to* $\mathfrak{P}$. *Then either* $\mathfrak{S}/\mathfrak{P}$ *is commutative,* $\mathcal{G}_1(\mathfrak{L}) \subseteq \mathfrak{P}$ *or* $\mathcal{G}_2(\mathfrak{L}) \subseteq \mathfrak{P}$.

Proof.
Assume that,

$$\mathcal{G}_1[\check{\imath}_1, \check{\imath}_2] - [\mathcal{G}_2(\check{\imath}_1), \check{\imath}_2] - [\check{\imath}_1, \mathcal{G}_1(\check{\imath}_2)] \in \mathfrak{P} \text{ for all } \check{\imath}_1, \check{\imath}_2 \in \mathfrak{L}. \tag{2.5}$$

When $\check{\imath}_1$ and $\check{\imath}_2$ are equal in Equation 2.5, the statement $[\mathcal{G}_2(\check{\imath}_1), \check{\imath}_1] + [\check{\imath}_1, \mathcal{G}_1(\check{\imath}_1)] \in \mathfrak{P}$ is true. Substitute $\check{\imath}_1\check{\imath}_2$ instead of $\check{\imath}_2$ in Equation 2.5 and use the last equation to conclude that

$$\mathcal{G}_1(\check{\imath}_1)\mathcal{G}_1([\check{\imath}_1, \check{\imath}_2]) - [\check{\imath}_1, \mathcal{G}_1(\check{\imath}_1)]\mathcal{G}_1(\check{\imath}_2) - \mathcal{G}_1(\check{\imath}_1)[\check{\imath}_1, \mathcal{G}_1(\check{\imath}_2)] \in \mathfrak{P}, \text{ for all } \check{\imath}_1, \check{\imath}_2 \in \mathfrak{L}. \tag{2.6}$$

Therefore, substituting Equation 2.5 into Equation 2.6 yields $\mathcal{G}_1(\check{\imath}_1)[\mathcal{G}_2(\check{\imath}_1), \check{\imath}_2] - [\check{\imath}_1, \mathcal{G}_1(\check{\imath}_1)]\mathcal{G}_1(\check{\imath}_2) \in \mathfrak{P}$, for all $\check{\imath}_1, \check{\imath}_2 \in \mathfrak{L}$. Replace $\check{\imath}_2$ with $\check{\imath}_2\mathcal{G}_2(\check{\imath}_1)$ in this equation, and then use it to fulfill $[\check{\imath}_1, \mathcal{G}_1(\check{\imath}_1)](\mathcal{G}_1(\check{\imath}_2) + \check{\imath}_2)\mathcal{G}_1(\mathcal{G}_2(\check{\imath}_1))$ for all $\check{\imath}_1, \check{\imath}_2 \in \mathfrak{L}$. Replacing $\check{\imath}_2$ in this equation with $\sum_{i=1}^{n}(-1)^{i-1}\mathcal{G}_1^{i-1}(\check{\imath}_2)$, where $\mathcal{G}_1^0(\mathfrak{S})$ is the identity mapping $id_{\mathfrak{S}}$. Since $\mathcal{G}_1$ is a Z-PV mapping on $\mathfrak{L}$, there exists an integer $\aleph(\check{\imath}) > 1$ for which $\mathcal{G}^{\aleph(\check{\imath})}(\check{\imath}) = 0$ for every $\check{\imath} \in \mathfrak{P}$. Thus $[\check{\imath}_1, \mathcal{G}_1(\check{\imath}_1)]\check{\imath}_2\mathcal{G}_1(\mathcal{G}_2(\check{\imath}_1)) \in \mathfrak{P}$ for all $\check{\imath}_1, \check{\imath}_2 \in \mathfrak{L}$. By utilizing [7, Fact 1] on the last equation, we deduce that either $[\check{\imath}_1, \mathcal{G}_1(\check{\imath}_1)] \in \mathfrak{P}$ or $\mathcal{G}_1(\mathcal{G}_2(\check{\imath}_1)) \in \mathfrak{P}$ for all $\check{\imath}_1 \in \mathfrak{L}$. Let $\mathcal{M}_1 = \{\omega \in \mathfrak{L} : [\omega, \mathcal{G}_1(\omega)] \in \mathfrak{P}\}$ and $\mathcal{M}_2 = \{\omega \in \mathfrak{L} : \mathcal{G}_1\mathcal{G}_2(\omega) \in \mathfrak{P}\}$. It is clear that $\mathfrak{L} = \mathcal{M}_1 \cup \mathcal{M}_2$. Hence using Brauer's trick fulfills either $\mathfrak{L} = \mathcal{M}_1$ or $\mathfrak{L} = \mathcal{M}_2$. From Theorem 2.1, the first case leads to either $\mathfrak{S}/\mathfrak{P}$ is commutative or $\mathcal{G}_1(\mathfrak{L}) \subseteq \mathfrak{P}$. The second case implies that the statement $\mathcal{G}_1\mathcal{G}_2(\omega) \in \mathfrak{P}$ holds for all $\omega \in \mathfrak{L}$. By applying Lemma 2.6, this case implies that either $\mathcal{G}_1(\mathfrak{L}) \subseteq \mathfrak{P}$ or $\mathcal{G}_2(\mathfrak{L}) \subseteq \mathfrak{P}$. □

The following corollary follows by taking $\mathcal{G}_1 = \mathcal{G}_2$ in the previous theorem
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Corollary 2.8. *Let* $\mathcal{G}_1$ *be an HD mapping of* $\mathfrak{S}$ *and a Z-PV on* $\mathfrak{L}$ *that satisfies for all* $\breve{\imath}_1, \breve{\imath}_2 \in \mathfrak{L}$*, the condition* $[\mathcal{G}_1(\breve{\imath}_1), \mathcal{G}_1(\breve{\imath}_2)]$ *belongs to* $\mathfrak{P}$*. Then either* $\mathfrak{S}/\mathfrak{P}$ *is commutative or* $\mathcal{G}_1(\mathfrak{L}) \subseteq \mathfrak{P}$*.*

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