International Journal of Mathematics and
Computer Science, **19**(2024), no. 4, 1231–1236

Distance Independence Polynomial of Graphs

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(Received January 17, 2024, Accepted May 21, 2024,
Published June 1, 2024)

Abstract

In this paper, we examine properties of points in a discrete structure by considering the independent sets with respect to proximity of points. Moreover, we represent properties in a bivariate polynomial which counts the number of subsets with a given proximity property.

1 Introduction

The study of graph representations in terms of polynomials captured the interests of discrete mathematicians because of their contributions in the area of Biology, Physics and Chemistry [4]. Independence polynomial was investigated by Hoede and Li [6] in 1994. This polynomial represents the

Key words and phrases: Independent set, distance independence polynomial.
AMS (MOS) Subject Classifications: 05C25, 05C30, 05C31.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

independent subsets in a graph. Many studies had been explored by considering substructures of a given graph.

An interesting work by Maldo and Artes [7] explored on the geodetic independence polynomial of graphs and used the Chuh-Shih-Chieh's Identity in establishing the polynomials. Recent work of Artes et al. [2] established some properties of polynomials representing geodetic closures. Also, Villarta et al. [8] pioneered the work on induced path polynomials and established results for graphs under some binary operations.

In this present work, we introduce a polynomial representation as a bivariate polynomial by considering the distribution property of vertices in a given independent set.

A subset S of $V(G)$ is said to be *independent* in G if the elements of S are pairwise non-adjacent in G. This means that for any two vertices $u, v \in S$, $uv \notin E(G)$. In this case, we call S an *independent set* (with respect to graph G) [1].

2 Preliminary Notions

Distance independence in graphs is a generalization of independence concept in graphs. Let k be a natural number greater than 1. A subset S of $V(G)$ is a d_k-*independent set* in G if for every pair of distinct vertices $(u, v) \in S \times S$, $d_G(u, v) \geq k$. The d_k-*independence polynomial* of G is given by

$$I_k(G; x, y) = \sum_{i=2}^{\alpha_k(G)} \alpha_{ik}(G) x^i y^k = y^k \sum_{i=2}^{\alpha_k(G)} a_{ik}(G) x^i$$

where $\alpha_{ik}(G)$ is the number d_k-independent subsets of G of cardinality i. The *distance independence polynomial* of G is given by

$$\Gamma_{di}(G; x, y) = \sum_{k=2}^{diam(G)} \sum_{i=2}^{\alpha_k(G)} \alpha_{ik}(G) x^i y^k$$

where $\alpha_{ik}(G)$ is the number of d_k-independent subsets of G of cardinality i, $\alpha_k(G)$ is the d_k-independence number of G, and $diam(G)$ is the diameter of G.

Hence, we have

$$\Gamma_{di}(G; x, y) = \sum_{k=2}^{diam(G)} I_k(G; x, y).$$

If the diameter of G is 2, then $\Gamma_{di}(G;x,y) = I_2(G;x,y) = y^2 I(G;x)$, where $I(G;x)$ is the independence polynomial of G.

The readers may refer to Harary [5] for some concepts not defined in this study.

3 Results

First, we consider the star $K_{1,n}$ of order $n+1$. This graph has a very nice property since the vertices except for the central vertex are distributed uniformly in terms of distances.

Theorem 3.1. *Let n be a natural number greater than 1. Then*

$$\Gamma_{di}(K_{1,n};x,y) = (1+x)^n y^2 - nxy^2 - y^2.$$

Proof. Let $K_{1,n}$ be the join of K_1 and $\overline{K_n}$, where $V(K_1) = \{u\}$ and $V(\overline{K_n}) = \{v_1, v_2, \ldots, v_n\}$. Let S be an independent set in $K_{1,n}$.

Case 1: $u \in S$.

In this case, $S = \{u\}$ since $uv \in E(K_{1,n})$ for every $v \in V(\overline{K_n})$.

Case 2: $u \notin S$.

Then S is a subset of $V(\overline{K_n})$. Let $v, w \in S$ with $v \neq w$. Then $d_{K_{1,n}}(v,w) = 2$ since vu and uw are edges of $K_{1,n}$ and $[v,u,w]$ is the unique $v-w$ geodesic in $K_{1,n}$ because both vertices are pendant and $K_{1,n}$ is a tree. Hence, S is a d_2-independent. Thus,

$$\begin{aligned}
\Gamma_{di}(K_{1,n};x,y) &= I_2(G;x,y) \\
&= \sum_{i=2}^{\alpha_2(G)} \alpha_{ik}(K_{i,1}) x^i y^2 \\
&= y^2 \sum_{i=2}^{n} \binom{n}{i} x^i \\
&= y^2\left((1+x)^n - nx - 1\right) \\
&= (1+x)^n y^2 - nxy^2 - y^2
\end{aligned}$$

□

The following lemma characterizes the d_k-independent subsets of $K_{m,n}$.

Lemma 3.2. *Suppose that m and n are natural numbers such that both are at least 3. A subset S of $V(K_{m,n})$ is d_k-independent in $K_{m,n}$ if and only if $k=2$ and S does not intersect the partite sets at the same time.*

Proof. Assume that S is a d_k-independent set. The assertion that $k = 2$ follows from the fact that the diameter of $K_{m,n}$ is 2. Suppose S intersects $V(\overline{K_m})$ and $V(\overline{K_n})$. Take $w \in S \cap V(\overline{K_m})$ and $z \in S \cap V(\overline{K_n})$. Then $wz \in E(K_{m,n})$. This contradicts the independence of S.

Suppose that $k = 2$ and S does not intersect the partite sets at the same time. Then $S \subseteq V(\overline{K_m})$ or $S \subseteq V(\overline{K_n})$.
Case 1: $S \subseteq V(\overline{K_m})$ Let $w, z \in S$. Then there exists $v \in V(\overline{K_n})$ such that $[w, v, z]$ is a $w - z$ geodesic in $K_{m,n}$. Thus, $d_{K_{m,n}}(w, z) = 2$. Hence, S is d_2-independent in $K_{m,n}$.

Case 2: $S \subseteq V(\overline{K_n})$ Let $p, q \in S$. Then there exists $u \in V(\overline{K_m})$ such that $[p, u, q]$ is a $p - q$ geodesic in $K_{m,n}$. Thus, $d_{K_{m,n}}(p, q) = 2$. Hence, S is d_2-independent in $K_{m,n}$. □

The complete bipartite graphs consist of two independent subsets and partitions join together by adding edges that connects between them. We established their distance independence polynomial in the next result.

Theorem 3.3. *Suppose that m and n are natural numbers such that both are at least 3. Then*

$$\Gamma_{di}(K_{m,n}; x, y) = (1 + x)^m y^2 + (1 + x)^n y^2 - mxy^2 - nxy^2 - 2y^2.$$

Proof. Consider an independent set subset S in $K_{m,n}$. Now, $S \cap V(\overline{K_m}) = \varnothing$ or $S \cap V(\overline{K_n}) = \varnothing$. Hence, either $S \subseteq V(\overline{K_n})$ or $S \subseteq V(\overline{K_m})$. Take distinct vertices $w, z \in S$. Then $d_{K_{m,n}}(w, z) = 2$. Thus, S is a d_2-independent set.

Case 1: $S \subseteq V(\overline{K_m})$

For each $i \in \{1, 2, \ldots, m\}$, the set $V(\overline{K_m})$ has exactly $\binom{m}{i}$ independent subsets of cardinality i. This contributes $\sum_{i=2}^{m} \binom{m}{i} x^i y^2$ to the distance independent polynomial of $K_{m,n}$.

Case 2: $S \subseteq V(\overline{K_n})$

For each $j \in \{1, 2, \ldots, n\}$, the set $V(\overline{K_n})$ has exactly $\binom{n}{j}$ independent subsets of cardinality j. This contributes $\sum_{j=2}^{n} \binom{n}{j} x^j y^2$ to the distance independent polynomial of $K_{m,n}$.

Combining Case 1 and Cases 2, we have

$$\begin{aligned}
\Gamma_{di}(K_{m,n};x,y) &= \sum_{i=2}^{m}\binom{m}{i}x^iy^2+\sum_{j=2}^{n}\binom{n}{j}x^jy^2\\
&= y^2\left[\sum_{i=0}^{m}\binom{m}{i}x^i-\binom{n}{0}x^0-\binom{m}{1}x^1\right]+y^2\left[\sum_{j=0}^{n}\binom{n}{j}x^j-\binom{n}{0}x^0-\binom{n}{1}x^1\right]\\
&= y^2\left[(1+x)^m-mx-1\right]+y^2\left[(1+x)^n-nx-1\right]\\
&= (1+x)^my^2+(1+x)^ny^2-mxy^2-nxy^2-2y^2.
\end{aligned}$$

If $m=n$, then we have

$$\Gamma_{di}(K_{m,m};x,y)=2(1+x)^my^2-2mxy^2-2y^2.$$

$\square$

Acknowledgment. The authors would like to acknowledge the support of Bohol Island State University and Mindanao State University-Tawi-Tawi College of Technology and Oceanography for the resources provided which are instrumental in the conduct and completion of this piece of work.

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