International Journal of Mathematics and
Computer Science, **19**(2024), no. 4, 1117–1123

The Non-Adjacency Compatible Vertices Topology on Digraphs

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(Received March 2, 2024, Accepted April 16, 2024,
Published June 1, 2024)

Abstract

In this work, we present a novel topology related to directed graphs which is called the non-adjacency compatible vertices topology (NCV-topology) of a directed graph $\mathcal{G}_\theta$. A sub-basis family of this topology is generated on the set of vertices and formed by taking the non-adjacency vertices which arise (path of length two) in the same direction to each vertex. We investigate some properties on important and certain types of directed graphs. Our goal is to provide foundational methods for examining certain aspects of directed graphs through the topology of their corresponding non-adjacency-compatible vertices. The presence of the isolated vertex is not necessary in our definition. We obtain some results on this new topology with certain types of di-graphs (which types of di-graphs achieve the discrete non-adjacency compatible vertices topology and which do not and consequently achieve T_1 and T_2-spaces).

Keywords and phrases: Directed graphs, NCV-topology, NCV-topological spaces.
AMS (MOS) Subject Classifications: 05C62, 05C75,05C99, 54A05.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

1 Introduction

Numerous applications have utilized the relationship between topology and graphs to generate many new types of topology generated by graphs. This shows the importance of topological graph theory which is part of graph theory which has a great role and illustrious history in mathematics [1]. On the basis of vertices or edges, some topological models are developed or based. In the directed and undirected graphs, Amiri et al. [2] began utilizing a graphic topology for each locally finite graph without an isolated vertex. In 2018, Abdu and Kilicman [3] defined the compatible edges topology induced by a sub-basis family on the edge set of directed graphs as a collection of sets of adjacent edges with each edge and construct a path of length two. In 2020, Hassan and Abed [4] introduced a new definition of the phrase "Family of Sub-basis" formed Independent Topology of any un-digraph (which could comprise one or more isolated vertices) via vertices that are not adjacent to the vertex $\gimel$. In 2022, Hassan and Zainy [5] presented the independent compatible edges topology and outlined it as the topology linked to the set $\mathbb{R}$ of edges via a non-adjacent edges which leads to a path of length three. In the same year, Ali and Hassan [6] defined an Independent Incompatible Edges topology based on di-graphs with some applications. In 2023, Hassan and Jaafar [7] introduced the non-incidence topology.

In Section 2, we give the main definitions of topologies and graphs. In Section 3, we define the non-adjacency compatible vertices topology (NCV-topology) associated with a digraph and give some examples. In Section 4, we conclude our paper.

2 Preliminaries

In this section, we cover some basic definitions and provide a brief introduction to graph theory and topologyu. These ideas are all commonly used and can be found in books like [1]. Usually the di-graph is a pair $\mathcal{G}_\theta = (\vartheta, \mathbb{R})$, where ϑ is the set of vertices and $\mathbb{R}$ is the set of directed edges. An edge of the form $\mathfrak{Z} = (u, u)$ is a loop. Parallel edges are those with the identical end vertices. If a graph contains no parallel edges or loop, then it is considered simple. If the vertices $\gimel$ and $\mathcal{U}$ are connected by an edge, then they are adjacent. We use the symbols K_n for the complete graph with n vertices, the symbol C_n for the cycle graph on n vertices, the symbol $\mathcal{P}_n$ for the path on n vertices, and the symbol $K_{n1,n2}$ for the whole bipartite graph of size partite n_1 and n_2. A topology T is a family of subsets that are open to the

non-empty set $\mathbb{F}$ if the following conditions hold:
i) $\mathbb{F}\phi \in T$,
ii)for every $H, M \in T$, $H \cap M \in T$,
iii) $\cup_{i\in\Delta}\ H_i \in T$ for every sub-combination H_i of T.
In this case, $(\mathbb{F}, T)$ is called a topological space. An indiscrete topology is defined as $T = \{\theta, \mathbb{F}\}$ on $\mathbb{F}$ while a discrete topology is defined as $T = P(\mathbb{F})$ on $\mathbb{F}$.

3 The Non-Adjacency Compatible Vertices Topological Space

In this section, we give the definition of a non-adjacency compatible vertices topological space (NCV-topological space) associated with a di-graph and provide some examples on a basic di-graph.

Definition 3.1. *Let $\mathcal{G}_\theta = (\vartheta, \mathbb{R})$ be any directed graph. The non-adjacency compatible vertices topology (NCV-topology) is a topology that relates to the set ϑ of vertices for $\mathcal{G}_\theta$, and brought on by sub-basis $\c{S}_{NCV}$ whose components are the sets $W \subseteq \vartheta, |W| \leq 2$. If $\daleth \in W$ and the vertex w is non-adjacent with vertex $\daleth$ such that $\daleth$ and w connected by path of length two in the same direction, then $w \in \mathcal{W}$.*

The set of vertices with $\mathcal{T}_{NCV}$ initiate the topological space $(\vartheta, \mathcal{T}_{NCV})$ which called non-adjacency compatible vertices topological space (NCV- topological space).

Example 3.2. *Consider the di-graph $G_\theta = (\vartheta, \mathbb{R})$ in Figure 1 below, such that $\vartheta(\mathcal{G}_\theta) = \{\daleth_1, \daleth_2, \daleth_3, \daleth_4, \daleth_5\}$, $\mathbb{R}(\mathcal{G}_\theta) = \{\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4, \mathcal{Z}_5\}$.*

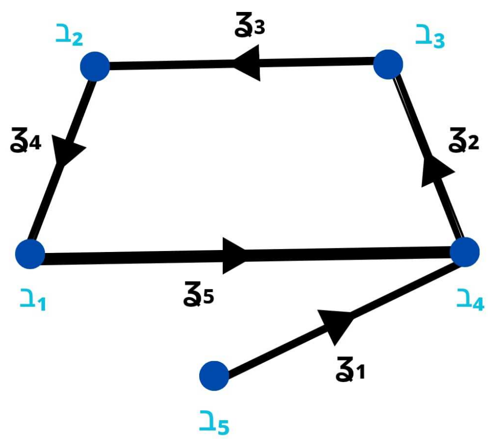

Figure 1: Simple directed graph

From the graph above, $\mathcal{T}_{NCV}$ *has a sub-basis* $Ş_{NCV} = \{\gimel_5, \gimel_3, \gimel_3, \gimel_1, \gimel_2, \gimel_4\}$*. Then by using finite intersection, the following base* β_{NCV} *is produced* $\{\{\gimel_5, \gimel_3\}, \{\gimel_3, \gimel_1\}, \{\gimel_2, \gimel_4\}, \gimel_3, \phi\}$*. Then, utilizing unions, we generate a topology* $\mathcal{T}_{NCV}$ *as follows:*

$$\mathcal{T}_{NCV} = \Big\{\phi, \vartheta, \{\gimel_5, \gimel_3\}, \{\gimel_3, \gimel_1\}, \{\gimel_2, \gimel_4\}, \{\gimel_3\}, \{\gimel_2, \gimel_4, \gimel_3\}, \{\gimel_1, \gimel_3, \gimel_5\}, \{\gimel_2, \gimel_4, \gimel_3, \gimel_1\}, \{\gimel_2, \gimel_4, \gimel_3, \gimel_5\}\Big\}$$

The items in the next remark follow directly from Definition 3.1.

Remark 3.3.

1. *The topology* $\mathcal{T}_{NCV}$ *on a di-graph* C_n *such that* $n = 4$ *is not a discrete topology, but when* $n > 4$ *is a discrete topology.*

2. *For every directed path* $\mathcal{P}_n$*, it is conceived that* $\mathcal{T}_{NCV}$ *is not a discrete topology.*

3. *For Every directed tree,* $\mathcal{T}_{NCV}$ *is not a discrete topology because the tree has pendant vertices which do not represent singleton sets.*

4. *The topology* $\mathcal{T}_{NCV}$ *of a complete bipartite di-graph* $K_{n1,n2}$ *is a discrete topology if* $n_1 \geq 3$ *and* $n_2 \geq 3$*.*

5. *The topology $\mathcal{T}_{NCV}$ on complete di-graph K_n is a discrete topology (when all edges are in the same direction)*

Proof:

To clarify (3) in Remark 3.3 above, we consider an example of a directed tree because the tree has pendant vertices which do not represent singleton sets, as in the example below:

Example 3.4. *Consider the di-graph $\mathcal{G}_\theta = (\vartheta, \mathbb{R})$ which a directed tree, in Fig. 2 shown such that:*
$\vartheta(\mathcal{G}_\theta) = \{\beth_1, \beth_2, \beth_3, \beth_4, \beth_5, \beth_6, \beth_7\}$, $\mathbb{R}\,(\mathcal{G}_\theta) = \{\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4, \mathcal{Z}_5, \mathcal{Z}_6\}$.

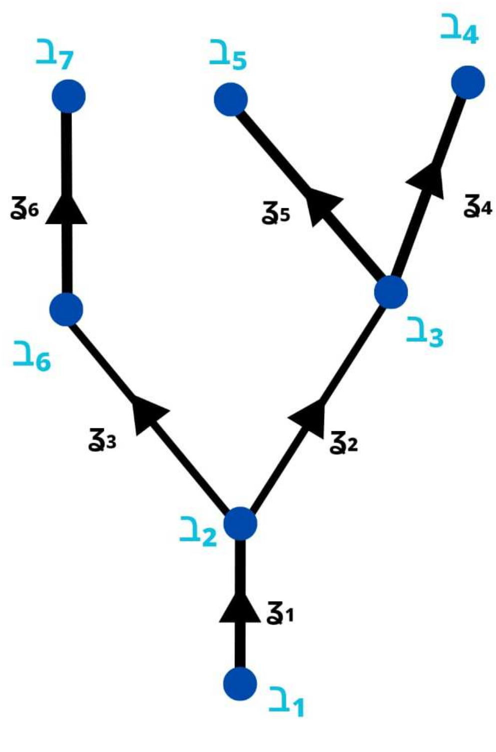

Figure 2: Directed tree

From the graph above, $\mathcal{T}_{NCV}$ *has a sub-basis* $Ş_{NCV} = \{\{\gimel_1, \gimel_3\}, \{\gimel_1, \gimel_6\}, \{\gimel_2, \gimel_7\}, \{\gimel_2, \gimel_4\}, \{\gimel_2, \gimel_5\}\}$. *Then, via finite intersection, the following base* β_{NCV} *is produced* $\{\phi, \{\gimel_1\}, \{\gimel_2\}, \{\gimel_1, \gimel_3\}, \{\gimel_1, \gimel_6\}, \{\gimel_2, \gimel_7\}, \{\gimel_2, \gimel_4\}, \{\gimel_2, \gimel_5\}\}$. *Then, utilizing unions, we generate a topology* $\mathcal{T}_{NCV}$ *as follows:*

$$\begin{aligned}
\mathcal{T}_{NCV} = \big\{ & \phi, \vartheta, \{\gimel_1\}, \{\gimel_2\}, \ \{\gimel_1, \gimel_3\}, \{\gimel_1, \gimel_6\}, \{\gimel_2, \gimel_7\}, \{\gimel_2, \gimel_4\}, \{\gimel_2, \gimel_5\}, \{\gimel_1, \gimel_2\}, \\
& \{\gimel_1, \gimel_2, \gimel_7\}, \{\gimel_1, \gimel_2, \gimel_4\}, \{\gimel_1, \gimel_2, \gimel_5\}, \{\gimel_1, \gimel_2, \gimel_6\}, \{\gimel_1, \gimel_3, \gimel_6\}, \{\gimel_2, \gimel_5, \gimel_7\}, \\
& \{\gimel_1, \gimel_2, \gimel_3\}, \{\gimel_4, \gimel_2, \gimel_7\}, \{\gimel_4, \gimel_2, \gimel_5\}, \{\gimel_1, \gimel_2, \gimel_3, \gimel_7\}, \{\gimel_1, \gimel_3, \gimel_2, \gimel_4\}, \\
& \{\gimel_1, \gimel_2, \gimel_3, \gimel_5\}, \{\gimel_1, \gimel_2, \gimel_6, \gimel_7\}, \{\gimel_1, \gimel_2, \gimel_4, \gimel_6\}, \{\gimel_1, \gimel_2, \gimel_5, \gimel_6\}, \\
& \{\gimel_1, \gimel_2, \gimel_4, \gimel_5\}, \{\gimel_1, \gimel_2, \gimel_4, \gimel_7\}, \{\gimel_2, \gimel_4, \gimel_5, \gimel_7\}, \{\gimel_1, \gimel_2, \gimel_3, \gimel_6, \gimel_7\}, \\
& \{\gimel_1, \gimel_2, \gimel_3, \gimel_4, \gimel_6\}, \{\gimel_1, \gimel_2, \gimel_3, \gimel_5, \gimel_6\}, \{\gimel_1, \gimel_2, \gimel_3, \gimel_4, \gimel_7\}, \{\gimel_1, \gimel_2, \gimel_3, \gimel_4, \gimel_5\}, \\
& \{\gimel_1, \gimel_2, \gimel_4, \gimel_6, \gimel_7\}, \{\gimel_1, \gimel_2, \gimel_4, \gimel_5, \gimel_6\}, \{\gimel_1, \gimel_2, \gimel_5, \gimel_6, \gimel_7\}, \{\gimel_1, \gimel_2, \gimel_4, \gimel_5, \gimel_7\}, \\
& \{\gimel_2, \gimel_3, \gimel_4, \gimel_5, \gimel_7\}, \{\gimel_1, \gimel_2, \gimel_3, \gimel_4, \gimel_5, \gimel_7\}, \{\gimel_1, \gimel_2, \gimel_3, \gimel_4, \gimel_6, \gimel_7\}, \\
& \{\gimel_1, \gimel_2, \gimel_3, \gimel_5, \gimel_6, \gimel_7\}, \{\gimel_1, \gimel_2, \gimel_3, \gimel_4, \gimel_6, \gimel_7\}, \{\gimel_1, \gimel_2, \gimel_3, \gimel_4, \gimel_5, \gimel_6\}, \\
& \{\gimel_1, \gimel_2, \gimel_4, \gimel_5, \gimel_6, \gimel_7\}\big\}
\end{aligned}$$

Clearly, $\mathcal{T}_{NCV}$ *is not a discrete topology because it does not have all the singletons.*

Proposition 3.5. *The topological space* $(\vartheta, \mathcal{T}_{NCV})$ *is a* T_1*-space If and only if* $\mathcal{T}_{NCV}$ *is a discrete topology.*

Proof:
This follows directly from the fact that any topological space $(\mathbb{F}, \mathcal{T})$ is a T_1-space if and only if $\mathcal{T} = P(\mathbb{F})$).

Proposition 3.6. *Let* $\mathcal{G}_\theta = (\vartheta, \mathbb{R})$ *be any directed graph and let* $\mathcal{T}_{NCV}$ *be a discrete on* $\mathcal{G}_\theta$*. Then* $(\vartheta, \mathcal{T}_{NCV})$ *is a* T_2*-space* $\Leftrightarrow$ $(\vartheta, \mathcal{T}_{NCV})$ *is a* T_1*-space.*

Proof:
$\Longrightarrow$) Obvious.
$\Longleftarrow$) Assume $(\vartheta, \mathcal{T}_{NCV})$ is a T_1-space. Then $(\vartheta, \mathcal{T}_{NCV})$ is discrete by Proposition 3.5. This implies that $\forall\ \gimel \in \vartheta, \{v\} \in \mathcal{T}_{NCV}$. So $\forall\ \gimel, w \in \vartheta$ such that $\gimel \ \neq w, \exists\ \{v\}, \{w\} \in \mathcal{T}_{NCV}$ such that $w \in \{w\}$ and $\gimel \in \{\gimel\}$ and $\{w\} \cap \{v\} = \emptyset$. Consequently, $(\vartheta, \mathcal{T}_{NCV})$ is a T_2-space

4 Conclusion

We presented a definition of a novel topology on directed graphs which we called the non-adjacency compatible vertices topological space (NCV-topological space) on the set of vertices. Moreover, we showed a few of the properties of this topology in some remarks and examples of basic di-graph types. Furthermore, we obtained some results dealing with an isolated vertex, the discrete non-adjacency compatible vertices topology, and T_1 and T_2-space).

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