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On characterizations of (τ_1, τ_2)-normal spaces

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Abstract

In this paper, we introduce the notion of (τ_1, τ_2)-normal spaces. Moreover, we investigate some characterizations of (τ_1, τ_2)-normal spaces.

1 Introduction

In 1971, Viglino [15] introduced the notion of seminormal spaces. Singal and Arya [12] introduced the class of almost normal spaces and proved that a space is normal if and only if it is both a seminormal space and an almost normal space. Paul and Bhattacharyya [11] introduced and studied the notion of p-normal spaces. Maheshwari and Prasad [10] introduced and

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investigated the concept of s-normal spaces. Buadong et al. [5] introduced and investigated new separation axioms in generalized topology and minimal structure spaces. Srisarakham and Boonpok [13] introduced some weak separation axioms by utilizing $\delta p(\Lambda, s)\mathscr{D}$-sets. In [2], the present authors studied some properties of (Λ, sp)-open sets. Boonpok and Viriyapong [1] introduced and investigated some weak separation axioms by utilizing the notions of (Λ, sp)-open sets and the (Λ, sp)-closure operator. Ekici [6] introduced a new class of spaces, called γ-normal spaces and studied the relationships among s-normal spaces, p-normal spaces and γ-normal spaces. In 2006, Ekici and Noiri [7] introduced and investigated the notions of δp-normal spaces, almost δp-normal spaces and mildly δp-normal spaces. On the other hand, the notions of $\mu_{(m,n)}$-normal spaces, (Λ, p)-normal spaces and $S\Lambda_s$-normal spaces were introduced by Torton et al. [14]; Khampakdee and Boonpok [9]; Khampakdee and Boonpok [8], respectively. In this paper, we introduce the notion of (τ_1, τ_2)-normal spaces. We also discuss some characterizations of (τ_1, τ_2)-normal spaces.

2 Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [4] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [4] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [4] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 2.1. [4] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2$-Cl(A) and $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Cl$(A)) = \tau_1\tau_2$-Cl(A).

(2) If $A \subseteq B$, then $\tau_1\tau_2$-Cl$(A) \subseteq \tau_1\tau_2$-Cl$(B)$.

(3) $\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl(A).

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [18] (resp. $(\tau_1, \tau_2)s$-*open* [3], $(\tau_1, \tau_2)p$-*open* [3], $(\tau_1, \tau_2)\beta$-*open* [3]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$-*open* [17] if $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is called $\alpha(\tau_1, \tau_2)$-*closed*.

3 Characterizations of (τ_1, τ_2)-normal spaces

In this section, we introduce the concept of (τ_1, τ_2)-normal spaces. Moreover, we discuss several characterizations of (τ_1, τ_2)-normal spaces.

Recall that a subset A of a bitopological space (X, τ_1, τ_2) is said to be *generalized* (τ_1, τ_2)-*closed* (briefly, g-(τ_1, τ_2)-closed) [16] if $\tau_1\tau_2\text{-Cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_1\tau_2$-open. A subset A is called g-(τ_1, τ_2)-*open* [16] if $X - A$ is g-(τ_1, τ_2)-closed.

Lemma 3.1. [16] *A subset A of a bitopological space (X, τ_1, τ_2) is g-(τ_1, τ_2)-open if and only if $F \subseteq \tau_1\tau_2\text{-}Int(A)$ whenever $F \subseteq A$ and F is $\tau_1\tau_2$-closed.*

Definition 3.2. *A bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-normal if for each pair of disjoint $\tau_1\tau_2$-closed sets F and F', there exist disjoint $\tau_1\tau_2$-open sets U and V such that $F \subseteq U$ and $F' \subseteq V$.*

Theorem 3.3. *For a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) (X, τ_1, τ_2) *is* (τ_1, τ_2)-*normal;*

(2) for each pair of disjoint $\tau_1\tau_2$-closed sets F and F', there exist disjoint g-(τ_1, τ_2)-open sets U and V such that $F \subseteq U$ and $F' \subseteq V$;

(3) for each $\tau_1\tau_2$-closed set F and each $\tau_1\tau_2$-open set G containing F, there exists a g-(τ_1, τ_2)-open set U such that $F \subseteq U \subseteq \tau_1\tau_2\text{-}Cl(U) \subseteq G$;

(4) for each $\tau_1\tau_2$-closed set F and each g-(τ_1, τ_2)-open set G containing F, there exists a $\tau_1\tau_2$-open set U such that

$$F \subseteq U \subseteq \tau_1\tau_2\text{-}Cl(U) \subseteq \tau_1\tau_2\text{-}Int(G);$$

(5) for each $\tau_1\tau_2$-closed set F and each g-(τ_1, τ_2)-open set G containing F, there exists a g-(τ_1, τ_2)-open set U such that

$$F \subseteq U \subseteq \tau_1\tau_2\text{-}Cl(U) \subseteq \tau_1\tau_2\text{-}Int(G);$$

(6) for each g-(τ_1, τ_2)-closed set F and each $\tau_1\tau_2$-open set G containing F, there exists a $\tau - 1\tau_2$-open set U such that

$$\tau_1\tau_2\text{-}Cl(F) \subseteq U \subseteq \tau_1\tau_2\text{-}Cl(U) \subseteq G;$$

(7) for each g-(τ_1, τ_2)-closed set F and each $\tau_1\tau_2$-open set G containing F, there exists a g-(τ_1, τ_2)-open set U such that

$$\tau_1\tau_2\text{-}Cl(F) \subseteq U \subseteq \tau_1\tau_2\text{-}Cl(U) \subseteq G.$$

Proof. (1) $\Rightarrow$ (2): The proof is obvious.

(2) $\Rightarrow$ (3): Let F be a $\tau_1\tau_2$-closed set and G be a (τ_1, τ_2)-open set containing F. Then, F and $X - G$ are two disjoint $\tau_1\tau_2$-closed sets. By (2), there exist disjoint g-(τ_1, τ_2)-open sets U and V such that $F \subseteq U$ and $X - G \subseteq V$. Since V is g-(τ_1, τ_2)-open and $X - G$ is $\tau_1\tau_2$-closed, by Lemma 3.1, $X - G \subseteq \tau_1\tau_2\text{-Int}(V)$. Since $U \cap V = \emptyset$, we have

$$\tau_1\tau_2\text{-Cl}(U) \subseteq \tau_1\tau_2\text{-Cl}(X - V) = X - \tau_1\tau_2\text{-Int}(V) \subseteq G.$$

Thus, $F \subseteq U \subseteq \tau_1\tau_2\text{-Cl}(U) \subseteq G$.

(3) $\Rightarrow$ (1): Let F and F' be any two disjoint $\tau_1\tau_2$-closed sets. Then, we have $X - F'$ is a $\tau_1\tau_2$-open set containing F. Thus by (3), there exists a g-(τ_1, τ_2)-open set U such that $F \subseteq U \subseteq \tau_1\tau_2\text{-Cl}(U) \subseteq X - F'$ and hence $F' \subseteq X - \tau_1\tau_2\text{-Cl}(U)$. Since F is $\tau_1\tau_2$-closed and U is g-(τ_1, τ_2)-open, by Lemma 3.1, we have $F \subseteq \tau_1\tau_2\text{-Int}(U)$. This shows that (X, τ_1, τ_2) is (τ_1, τ_2)-normal.

(6) $\Rightarrow$ (7) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (5): Let F be a $\tau_1\tau_2$-closed set and G be a g-(τ_1, τ_2)-open set containing F. Since G is g-(τ_1, τ_2)-open and F is $\tau_1\tau_2$-closed, by Lemma 3.1, $F \subseteq \tau_1\tau_2\text{-Int}(G)$. Thus by (3), there exists a g-(τ_1, τ_2)-open set U such that $F \subseteq U \subseteq \tau_1\tau_2\text{-Cl}(U) \subseteq \tau_1\tau_2\text{-Int}(G)$.

(5) $\Rightarrow$ (6): Let F be a g-(τ_1, τ_2)-closed set and G be a $\tau_1\tau_2$-open set containing F. Then, we have $\tau_1\tau_2\text{-Cl}(F) \subseteq G$. Since G is g-(τ_1, τ_2)-open and by (5), there exists a g-(τ_1, τ_2)-open set U such that

$$\tau_1\tau_2\text{-Cl}(F) \subseteq U \subseteq \tau_1\tau_2\text{-Cl}(U) \subseteq G.$$

Since U is g-(τ_1, τ_2)-open and $\tau_1\tau_2$-Cl(F) is $\tau_1\tau_2$-closed, by Lemma 3.1,

$$\tau_1\tau_2\text{-Cl}(F) \subseteq \tau_1\tau_2\text{-Int}(U).$$

Put $V = \tau_1\tau_2$-Int(U). Then, we have V is $\tau_1\tau_2$-open and

$$\tau_1\tau_2\text{-Cl}(F) \subseteq V \subseteq \tau_1\tau_2\text{-Cl}(V) = \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(U)) \subseteq \tau_1\tau_2\text{-Cl}(U) \subseteq G.$$

(4) $\Rightarrow$ (5) $\Rightarrow$ (2): This is obvious.

(6) $\Rightarrow$ (4): Let F be a $\tau_1\tau_2$-closed set and G be a g-(τ_1, τ_2)-open set containing F. By Lemma 3.1, $F \subseteq \tau_1\tau_2$-Int(G). Since F is g-(τ_1, τ_2)-closed and $\tau_1\tau_2$-Int(G) is $\tau_1\tau_2$-open, by (6), there exists a $\tau_1\tau_2$-open set U such that $F = \tau_1\tau_2$-Cl$(F) \subseteq U \subseteq \tau_1\tau_2$-Cl$(U) \subseteq \tau_1\tau_2$-Int$(G)$. □

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