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On characterizations of (τ_1, τ_2)-regular spaces

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Abstract

In this paper, we introduce the notion of (τ_1, τ_2)-regular spaces and investigate some characterizations of such spaces.

1 Introduction

It is well known that various types of separation axioms play a significant role in the theory of classical point set topology. In literature, separation axioms have been studied by many mathematicians. Sinnal and Arya [12] defined a new separation axiom called almost regularity which is weaker than regularity. Mashhour et al [9] introduced and investigated the concept of preopen

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sets and preclosed sets in topological spaces. El-Deeb et al. [6] introduced and studied the notion of p-regular spaces by using preopen sets. Malghan and Navalagi [8] introduced and investigated the concept of almost p-regular spaces as a generalization of p-regularity. Noiri [10] defined a new class of sets called rgp-closed sets and investigated some characterizations of almost p-regular spaces by utilizing rgp-closed sets. Buadong et al. [5] introduced and investigated new separation axioms in generalized topology and minimal structure spaces. Srisarakham and Boonpok [13] introduced some weak separation axioms by utilizing $\delta p(\Lambda, s)\mathscr{D}$-sets. In [2], the present authors studied some properties of (Λ, sp)-open sets. Boonpok and Viriyapong [1] introduced and investigated some weak separation axioms by utilizing the notions of (Λ, sp)-open sets and the (Λ, sp)-closure operator. Torton et al. [14] introduced and studied the notion of $\mu_{(m,n)}$-regular spaces. Furthermore, several characterizations of (Λ, p)-regular spaces and $S\Lambda_s$-regular spaces were presented in [11] and [7], respectively. In this paper, we introduce the notion of (τ_1, τ_2)-regular spaces. Moreover, some characterizations of (τ_1, τ_2)-regular spaces are discussed.

2 Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [4] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [4] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [4] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 2.1. [4] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2$-$Cl(X - A) = X - \tau_1\tau_2$-$Int(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [17] (resp. $(\tau_1, \tau_2)s$-*open* [3], $(\tau_1, \tau_2)p$-*open* [3], $(\tau_1, \tau_2)\beta$-*open* [3]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))))$. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$-*open* [16] if $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is called $\alpha(\tau_1, \tau_2)$-*closed*.

3 Characterizations of (τ_1, τ_2)-regular spaces

In this section, we introduce the notion of $(\tau_1, p\tau_2)$-regular spaces. Moreover, we discuss some characterizations of (τ_1, τ_2)-regular spaces.

Recall that a subset A of a bitopological space (X, τ_1, τ_2) is said to be *generalized* (τ_1, τ_2)-*closed* (briefly, g-(τ_1, τ_2)-closed) [15] if $\tau_1\tau_2$-Cl$(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_1\tau_2$-open. A subset A is called g-(τ_1, τ_2)-*open* [15] if $X - A$ is g-(τ_1, τ_2)-closed.

Lemma 3.1. [15] *A subset A of a bitopological space (X, τ_1, τ_2) is g-(τ_1, τ_2)-open if and only if $F \subseteq \tau_1\tau_2$-$Int(A)$ whenever $F \subseteq A$ and F is $\tau_1\tau_2$-closed.*

Definition 3.2. *A bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-regular if for each $\tau_1\tau_2$-closed set F and each point $x \in X - F$, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $x \in U$ and $F \subseteq V$.*

Theorem 3.3. *For a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) *(X, τ_1, τ_2) is (τ_1, τ_2)-regular.*

(2) *For each $x \in X$ and each $\tau_1\tau_2$-open set U with $x \in U$, there exists a $\tau_1\tau_2$-open set V such that $x \in V \subseteq \tau_1\tau_2$-$Cl(V) \subseteq U$.*

(3) *For each $\tau_1\tau_2$-closed set F of X,*

$$\cap\{\tau_1\tau_2\text{-}Cl(V) \mid F \subseteq V \text{ and } V \text{ is } \tau_1\tau_2\text{-open}\} = F.$$

(4) *For each subset A of X and each $\tau_1\tau_2$-open set U with $A \cap U \neq \emptyset$, there exists a $\tau_1\tau_2$-open set V such that $A \cap V \neq \emptyset$ and $\tau_1\tau_2$-$Cl(V) \subseteq U$.*

(5) *For each nonempty subset A of X and each $\tau_1\tau_2$-closed set F of X with $A \cap F = \emptyset$, there exist $\tau_1\tau_2$-open sets V and W such that $A \cap V \neq \emptyset$, $F \subseteq W$ and $V \cap W = \emptyset$.*

(6) *For each $\tau_1\tau_2$-closed set F of X and $x \notin F$, there exist a $\tau_1\tau_2$-open set U and a g-(τ_1, τ_2)-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$.*

(7) *For each subset A of X and each $\tau_1\tau_2$-closed set F with $A \cap F = \emptyset$, there exist a $\tau_1\tau_2$-open set U and a g-(τ_1, τ_2)-open set V such that $A \cap U \neq \emptyset$, $F \subseteq V$ and $U \cap V = \emptyset$.*

Proof. (1) $\Rightarrow$ (2): Let G be a $\tau_1\tau_2$-open set and $x \notin X - G$. Then, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $X - G \subseteq U$ and $x \in V$. Thus, $V \subseteq X - U$ and hence $x \in V \subseteq \tau_1\tau_2\text{-Cl}(V) \subseteq X - U \subseteq G$.

(2) $\Rightarrow$ (3): Let F be a $\tau_1\tau_2$)-closed set and $x \notin X - F$. By (2), there exists a $\tau_1\tau_2$-open set U such that $x \in U \subseteq \tau_1\tau_2\text{-Cl}(U) \subseteq X - F$. Therefore, $F \subseteq X - \tau_1\tau_2\text{-Cl}(U) = V$. Since V is $\tau_1\tau_2$-open and $U \cap V = \emptyset$. Thus, $x \notin \tau_1\tau_2\text{-Cl}(V)$ and hence $F \supseteq \cap\{\tau_1\tau_2\text{-Cl}(V) \mid F \subseteq V$ and V is $\tau_1\tau_2\text{-open}\}$.

(3) $\Rightarrow$ (4): Let A be a subset of X and U be a $\tau_1\tau_2$-open set such that $A \cap U \neq \emptyset$. Let $x \in A \cap U$. Then, we have $x \notin X - U$. Thus by (3), there exists a $\tau_1\tau_2$-open set W such that $X - U \subseteq W$ and $x \notin \tau_1\tau_2\text{-Cl}(W)$. Put $V = X - \tau_1\tau_2\text{-Cl}(W)$ which is a $\tau_1\tau_2$-open set containing x and hence $A \cap V \neq \emptyset$. Now, $V \subseteq X - W$ and so $\tau_1\tau_2\text{-Cl}(V) \subseteq X - W \subseteq U$.

(4) $\Rightarrow$ (5): Let A be a nonempty subset of X and F be a $\tau_1\tau_2$-closed set such that $A \cap F = \emptyset$. Then, $X - F$ is $\tau_1\tau_2$-open and $A \cap (X - F) \neq \emptyset$. By (4), there exists a $\tau_1\tau_2$-open set V such that $A \cap V \neq \emptyset$ and $\tau_1\tau_2\text{-Cl}(V) \subseteq X - F$. If we put $W = X - \tau_1\tau_2\text{-Cl}(V)$, then $F \subseteq W$ and $W \cap V = \emptyset$.

(5) $\Rightarrow$ (1): Let F be a $\tau_1\tau_2$-closed set not containing x. Then, $F \cap \{x\} = \emptyset$. Thus by (5), there exist $\tau_1\tau_2$-open sets V and W such that $x \in V$, $F \subseteq W$ and $V \cap W = \emptyset$.

(1) $\Rightarrow$ (6): The proof is obvious.

(6) $\Rightarrow$ (7): Let A be a subset of X and F be a $\tau_1\tau_2$-closed set such that $A \cap F = \emptyset$. Then, for each $x \in A$, $x \notin F$. By (6), there exist a $\tau_1\tau_2$-open set U and a g-(τ_1, τ_2)-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$. Thus, $A \cap U \neq \emptyset$, $F \subseteq V$ and $U \cap V = \emptyset$.

(7) $\Rightarrow$ (1): Let F be a $\tau_1\tau_2$-closed set such that $x \notin F$. Since $\{x\} \cap F = \emptyset$, by (7) there exist a $\tau_1\tau_2$-open set U and a g-(τ_1, τ_2)-open set W such that $x \in U$, $F \subseteq W$ and $U \cap W = \emptyset$. Since W is g-(τ_1, τ_2)-open, by Lemma 3.1, $F \subseteq \tau_1\tau_2\text{-Cl}(W) = V$ and $U \cap V = \emptyset$. This shows that (X, τ_1, τ_2) is (τ_1, τ_2)-regular. □

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References

[1] C. Boonpok, C. Viriyapong, Some applications of (Λ, sp)-open sets in topological spaces, Eur. J. Pure Appl. Math., **15,** no. 3, (2022), 878–886.

[2] C. Boonpok, J. Khampakdee, (Λ, sp)-open sets in topological spaces, Eur. J. Pure Appl. Math., **15,** no. 2, (2022), 572–588.

[3] C. Boonpok, $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions, Heliyon, **6,** (2020), e05367.

[4] C. Boonpok, C. Viriyapong, M. Thongmoon, On upper and lower (τ_1, τ_2)-precontinuous multifunctions, J. Math. Comput. Sci., **18,** (2018), 282–293.

[5] S. Buadong, C. Viriyapong, C. Boonpok, On generalized topology and minimal structure spaces, Int. J. Math. Anal., **5,** no. 31, (2011), 1507–1516.

[6] N. El-Deeb, I. A. Hasanein, A. S. Mashhour, T. Noiri, On p-regular spaces, Bull. Math. Soc. Sci. Math. R. S. Roumanie, **27,** no. 75, (1983), 311–315.

[7] J. Khampakdee, C. Boonpok, On $S\Lambda_s$-regular spaces and $S\Lambda_s$-normal spaces, Int. J. Math. Comput. Sci., **19,** no. 1, (2024), 125–129.

[8] S. R. Malghan, G. B. Navalagi, Almost p-regular, p-completely regular and almost p-completely regular spaces, Bull. Math. Soc. Sci. Math. R. S. Roumanie, **34,** no. 82, (1990), 317–326.

[9] A. S. Mashhour, M. E. Abd El-Monsef, S. N. El-Deeb, On precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. Egypt, **53,** (1982), 47–53.

[10] T. Noiri, Almost p-regular spaces and some functions, Acta Math. Hungar., **79,** no. 3, (1998), 207–216.

[11] P. Pue-on, C. Boonpok, On $S\Lambda_s$-regular spaces and $S\Lambda_s$-normal spaces, Int. J. Math. Comput. Sci., **18,** no. 2, (2023), 295–299.

[12] M. K. Singal, S. P. Arya, On almost-regular spaces, Glasnik Mat., **4,** no. 24, (1969), 89–99.

[13] N. Srisarakham, C. Boonpok, On characterizations of $\delta p(\Lambda, s)$-$\mathscr{D}_1$ spaces, Int. J. Math. Comput. Sci., **18,** no. 4, (2023), 743–747.

[14] P. Torton, C. Viriyapong, C. Boonpok, Some separation axioms in bi-generalized topological spaces, Int. J. Math. Anal., **6,** no. 56, (2012), 2789–2796.

[15] C. Viriyapong, S. Sompong, C. Boonpok, Generalized (τ_1, τ_2)-closed sets in bitopological spaces, Int. J. Math. Comput. Sci., **19,** no. 3, (2024), 821–826.

[16] N. Viriyapong, S. Sompong, C. Boonpok, (τ_1, τ_2)-extremal disconnectedness in bitopological spaces, Int. J. Math. Comput. Sci., **19,** no. 3, (2024), 855–860.

[17] C. Viriyapong, C. Boonpok, $(\tau_1, \tau_2)\alpha$-continuity for multifunctions, J. Math., (2020), 6285763.