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$(\tau_1, \tau_2)\theta$-closed sets and weak (τ_1, τ_2)-continuity

Butsakorn Kong-ied[1], Supannee Sompong[2], Chawalit Boonpok[1]

[1]Mathematics and Applied Mathematics Research Unit
Department of Mathematics
Faculty of Science
Mahasarakham University
Maha Sarakham, 44150, Thailand

[2]Department of Mathematics and Statistics
Faculty of Science and Technology
Sakon Nakhon Rajbhat University
Sakon Nakhon, 47000, Thailand

email: butsakorn.k@msu.ac.th, s_sompong@snru.ac.th,
chawalit.b@msu.ac.th

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Abstract

In this paper, we investigate some characterizations of upper and lower weakly (τ_1, τ_2)-continuous multifunctions by utilizing the notion of $(\tau_1, \tau_2)\theta$-closed sets.

1 Introduction

In 1961, Levine [9] introduced and studied the notion of weakly continuous functions. Moreover, some characterizations of pairwise weakly M-continuous functions and weakly $(\mu, \mu')^{(m,n)}$-continuous functions were investigated in [6] and [7], respectively. Popa [15] and Smithson [16] independently

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The corresponding author is Butsakorn Kong-ied.
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introduced the notion of weakly continuous multifunctions. The present authors introduced and studied other weak forms of continuous multifunctions: weakly quasicontinuous multifunctions [12], almost weakly continuous multifunctions [11], weakly α-continuous multifunctions [14], weakly β-continuous multifunctions [13]. These multifunctions have similar characterizations. The analogy in their definitions and results suggests the need to formulate a unified theory. Noiri and Popa [10] introduced and studied the notions of upper and lower weakly m-continuous multifunctions as a multifunction from a set satisfying certain minimal condition into a topological space. Laprom et al. [8] introduced and studied the notion of $\beta(\tau_1,\tau_2)$-continuous multifunctions. Viriyapong and Boonpok [18] introduced and investigated the notion of weakly $(\tau_1,\tau_2)\alpha$-continuous multifunctions. Furthermore, several characterizations of weakly $(\tau_1,\tau_2)\delta$-semicontinuous multifunctions and almost weakly (τ_1,τ_2)-continuous multifunctions were studied in [4] and [3], respectively. In this paper, we investigate some characterizations of upper and lower weakly (τ_1,τ_2)-continuous multifunctions by utilizing $(\tau_1,\tau_2)\theta$-closed sets and $(\tau_1,\tau_2)\theta$-open sets.

2 Preliminaries

Throughout this paper, spaces (X,τ_1,τ_2) and (Y,σ_1,σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X,τ_1,τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i=1,2$. A subset A of a bitopological space (X,τ_1,τ_2) is called $\tau_1\tau_2$-*closed* [5] if $A=\tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [5] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [5] of A and is denoted by $\tau_1\tau_2$-Int(A).

A subset A of a bitopological space (X,τ_1,τ_2) is said to be $(\tau_1,\tau_2)r$-*open* [18] if $A=\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$. Let A be a subset of a bitopological space (X,τ_1,τ_2). A point $x\in X$ is called a $(\tau_1,\tau_2)\theta$-*cluster point* [18] of A if

$\tau_1\tau_2$-Cl$(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$-*closure* [18] of A and is denoted by $(\tau_1, \tau_2)\theta$-Cl(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$-*closed* [18] if $(\tau_1, \tau_2)\theta$-Cl$(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$-*open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\theta$-*interior* [18] of A and is denoted by $(\tau_1, \tau_2)\theta$-Int(A).

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [1] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3 Characterizations of upper and lower weakly (τ_1, τ_2)-continuous multifunctions

In this section, we investigate some characterizations of upper and lower (τ_1, τ_2)-continuous multifunctions.

Definition 3.1. [17] *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower weakly (τ_1, τ_2)-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $\sigma_1\sigma_2$-Cl$(V) \cap F(z) \neq \emptyset$ for each $z \in U$.*

Lemma 3.2. *If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower weakly (τ_1, τ_2)-continuous, then for each $x \in X$ and each subset B of Y with $(\sigma_1, \sigma_2)\theta$-Int$(B) \cap F(x) \neq \emptyset$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $U \subseteq F^-(B)$.*

Proof. Since $(\sigma_1, \sigma_2)\theta$-Int$(B) \cap F(x) \neq \emptyset$, there exists a nonempty $\sigma_1\sigma_2$-open set V of Y such that $\sigma_1\sigma_2$-Cl$(V) \subseteq B$ and $F(x) \cap V \neq \emptyset$. Since F is lower weakly (τ_1, τ_2)-continuous, there exists a $\tau_1\tau_2$-open set U of X containing x such that $\sigma_1\sigma_2$-Cl$(V) \cap F(z) \neq \emptyset$ for each $z \in U$ and hence $U \subseteq F^-(B)$. □

Lemma 3.3. [17] *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower weakly (τ_1, τ_2)-continuous;

(2) $F^-(V) \subseteq \tau_1\tau_2$-Int$(F^-(\sigma_1\sigma_2$-Cl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $\tau_1\tau_2$-*Cl*$(F^-(\sigma_1\sigma_2$-*Int*$(K))) \subseteq F^-(K)$ *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(4) $\tau_1\tau_2$-*Cl*$(F^+(\sigma_1\sigma_2$-*Int*$(\sigma_1\sigma_2$-*Cl*$(B)))) \subseteq F^+(\sigma_1\sigma_2$-*Cl*$(B))$ *for every subset* B *of* Y*;*

(5) $F^-(\sigma_1\sigma_2$-*Int*$(B)) \subseteq \tau_1\tau_2$-*Int*$(F^-(\sigma_1\sigma_2$-*Cl*$(\sigma_1\sigma_2$-*Int*$(B))))$ *for every subset* B *of* Y*;*

(6) $\tau_1\tau_2$-*Cl*$(F^+(\sigma_1\sigma_2$-*Int*$(\sigma_1\sigma_2$-*Cl*$(V)))) \subseteq F^+(\sigma_1\sigma_2$-*Cl*$(V))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(7) $\tau_1\tau_2$-*Cl*$(F^+(V)) \subseteq F^+(\sigma_1\sigma_2$-*Cl*$(V))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(8) $\tau_1\tau_2$-*Cl*$(F^+(\sigma_1\sigma_2$-*Cl*$(K))) \subseteq F^+(K)$ *for every* $(\sigma_1, \sigma_2)r$*-closed set* K *of* Y*.*

Theorem 3.4. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is lower weakly* (τ_1, τ_2)*-continuous;*

(2) $\tau_1\tau_2$-*Cl*$(F^+(B)) \subseteq F^+((\sigma_1, \sigma_2)\theta$-*Cl*$(B))$ *for every subset* B *of* Y*;*

(3) $F(\tau_1\tau_2$-*Cl*$(A)) \subseteq (\sigma_1, \sigma_2)\theta$-*Cl*$(F(A))$ *for every subset* A *of* X*.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Suppose that

$$x \notin F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(B)).$$

Then, we have $x \in F^-(Y - (\sigma_1, \sigma_2)\theta\text{-Cl}(B)) = F^-((\sigma_1, \sigma_2)\theta\text{-Int}(Y - B))$. By Lemma 3.2, there exists a $\tau_1\tau_2$-open set U of X containing x such that $U \subseteq F^-(Y - B) = X - F^+(B)$. Thus $U \cap F^+(B) = \emptyset$ and hence

$$x \notin \tau_1\tau_2\text{-Cl}(F^+(B)).$$

This shows that $\tau_1\tau_2\text{-Cl}(F^+(B)) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$.

(2) $\Rightarrow$ (3): Let A be any subset of X. By (2), we have $\tau_1\tau_2\text{-Cl}(A) \subseteq \tau_1\tau_2\text{-Cl}(F^+(F(A))) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(F(A)))$. Thus,

$$F(\tau_1\tau_2\text{-Cl}(A)) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(F(A)).$$

(3) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then $\sigma_1\sigma_2\text{-Cl}(V) = (\sigma_1, \sigma_2)\theta\text{-Cl}(V)$ and by (3), $F(\tau_1\tau_2\text{-Cl}(F^+(V))) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(F(F^+(V))) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(V) = \sigma_1\sigma_2\text{-Cl}(V)$. Thus, $\tau_1\tau_2\text{-Cl}(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$ and by Lemma 3.3, F is lower weakly (τ_1, τ_2)-continuous. □

Definition 3.5. [17] *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper weakly (τ_1, τ_2)-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$.*

Lemma 3.6. [17] *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper weakly (τ_1, τ_2)-continuous;

(2) $F^+(V) \subseteq \tau_1\tau_2$-Int$(F^+(\sigma_1\sigma_2$-Cl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $\tau_1\tau_2$-Cl$(F^-(\sigma_1\sigma_2$-Int$(K))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2$-Cl$(F^-(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(B)))) \subseteq F^-(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y;

(5) $F^+(\sigma_1\sigma_2$-Int$(B)) \subseteq \tau_1\tau_2$-Int$(F^+(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(B))))$ for every subset B of Y;

(6) $\tau_1\tau_2$-Cl$(F^-(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq F^-(\sigma_1\sigma_2$-Cl$(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(7) $\tau_1\tau_2$-Cl$(F^-(V)) \subseteq F^-(\sigma_1\sigma_2$-Cl$(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(8) $\tau_1\tau_2$-Cl$(F^-(\sigma_1\sigma_2$-Cl$(K))) \subseteq F^-(K)$ for every $(\sigma_1, \sigma_2)r$-closed set K of Y.

Theorem 3.7. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \tau_2)$, the following properties are equivalent:*

(1) F is upper weakly (τ_1, τ_2)-continuous;

(2) $\tau_1\tau_2$-Cl$(F^-(\sigma_1\sigma_2$-Int$((\sigma_1, \sigma_2)\theta$-Cl$(B)))) \subseteq F^-((\sigma_1, \sigma_2)\theta$-Cl$(B))$ for every subset B of Y;

(3) $\tau_1\tau_2$-Cl$(F^-(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(B)))) \subseteq F^-((\sigma_1, \sigma_2)\theta$-Cl$(B))$ for every subset B of Y.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Then $(\sigma_1, \sigma_2)\theta$-Cl$(B)$ is $\sigma_1\sigma_2$-closed in Y and by Lemma 3.6, $\tau_1\tau_2$-Cl$(F^-(\sigma_1\sigma_2$-Int$((\sigma_1, \sigma_2)\theta$-Cl$(B)))) \subseteq F^-((\sigma_1, \sigma_2)\theta$-Cl$(B))$.

(2) $\Rightarrow$ (3): The proof is obvious.

(3) $\Rightarrow$ (1): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then, we have $(\sigma_1, \sigma_2)\theta\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)) = \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)) = K$ and hence

$$\begin{aligned}\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K))) &= \tau_1\tau_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))\\ &\subseteq F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))\\ &= F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) = F^-(K).\end{aligned}$$

Thus, by Lemma 3.6, F is upper weakly (τ_1, τ_2)-continuous. □

Theorem 3.8. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower weakly (τ_1, τ_2)-continuous;

(2) $\tau_1\tau_2$-Cl$(F^+(\sigma_1\sigma_2$-Int$((\sigma_1, \sigma_2)\theta$-Cl$(B)))) \subseteq F^+((\sigma_1, \sigma_2)\theta$-Cl$(B))$ for every subset B of Y;

(3) $\tau_1\tau_2$-Cl$(F^+(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(B)))) \subseteq F^+((\sigma_1, \sigma_2)\theta$-Cl$(B))$ for every subset B of Y.

Proof. The proof is similar to that of Theorem 3.7. □

Definition 3.9. [2] *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly (τ_1, τ_2)-continuous if f has this property at each point of X.*

Corollary 3.10. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly (τ_1, τ_2)-continuous;

(2) $\tau_1\tau_2$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$((\sigma_1, \sigma_2)\theta$-Cl$(B)))) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta$-Cl$(B))$ for every subset B of Y;

(3) $\tau_1\tau_2$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(B)))) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta$-Cl$(B))$ for every subset B of Y.

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References

[1] C. Berge, Espaces topologiques fonctions multivoques, Dunod, Paris, 1959.

[2] C. Boonpok, C. Klanarong, On weakly (τ_1, τ_2)-continuous functions, Eur. J. Pure Appl. Math., **17,** no. 1, (2024), 416–425.

[3] C. Boonpok, C. Viriyapong, Upper and lower almost weak (τ_1, τ_2)-continuity, Eur. J. Pure Appl. Math., **14,** no. 4, (2021), 1212–1225.

[4] C. Boonpok, $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions, Heliyon, **6** (2020), e05367.

[5] C. Boonpok, C. Viriyapong and M. Thongmoon, On upper and lower (τ_1, τ_2)-precontinuous multifunctions, J. Math. Computer Sci., **18,** (2018), 282–293.

[6] C. Boonpok, M-continuous functions in biminimal structure spaces, Far East J. Math. Sci., **43,** no. 1, (2010), 41–58.

[7] T. Duangphui, C. Boonpok, C. Viriyapong, Continuous functions on bigeneralized topological spaces, Int. J. Math. Anal., **5,** no. 24, (2011), 1165–1174.

[8] K. Laprom, C. Boonpok, C. Viriyapong, $\beta(\tau_1, \tau_2)$-continuous multifunctions on bitopological spaces, J. Math., (2020), 4020971.

[9] N. Levine, A decomposition of continuity in topological spaces, Amer. Math. Monthly, **68,** (1961), 44–46.

[10] T. Noiri, V. Popa, A unified theory of weak continuity for multifunctions, Stud. Circ. St. Ser. Mat. Univ. Bacău, **16,** (2006), 167–200.

[11] T. Noiri, V. Popa, Almost weakly continuous multifunctions, Demonstratio Math., **26,** (1993), 363–380.

[12] T. Noiri, V. Popa, On upper and lower weakly quasicontinuous multifunctions, Rev. Roumaine Math. Pures Appl. **36,** (1992), 499–508.

[13] V. Popa, T. Noiri, On upper and lower weakly β-continuous multifunctions, Anal. Univ. Sci. Budapest, **43,** (2000), 25–48.

[14] V. Popa, T. Noiri, On upper and lower almost α-continuous multifunctions, Demonstratio Math., **29,** (1996), 381–396.

[15] V. Popa, Weakly continuous multifunctions, Boll. Un. Mat. Ital., (5) **15(A)** (1978), 379–388.

[16] R. E. Smithson, Almost and weak continuity for multifunctions, Bull. Calcutta Math. Soc., **70,** (1978), 383–390.

[17] M. Thongmoon, S. Sompong, C. Boonpok, Upper and lower weak (τ_1, τ_2)-continuity, (submitted).

[18] C. Viriyapong, C. Boonpok, $(\tau_1, \tau_2)\alpha$-continuity for multifunctions, J. Math., (2020), 6285763.