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(τ_1, τ_2)-continuity and weak (τ_1, τ_2)-continuity

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Abstract

In this paper, we investigate the relationships between (τ_1, τ_2)-continuous multifunctions and weakly (τ_1, τ_2)-continuous multifunctions.

1 Introduction

The notion of weakly continuous functions was introduced by Levine [11]. Popa and Noiri [12] introduced the notion of weakly (τ, m)-continuous functions as functions from a topological space into a set satisfying some minimal conditions and investigated several characterizations of weakly (τ, m)-continuous functions. Duangphui et al. [8] introduced and studied the notion

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of weakly $(\mu, \mu')^{(m,n)}$-continuous functions. Moreover, several characterizations of pairwise weakly M-continuous functions were presented in [5]. Popa [15] and Smithson [17] independently introduced the notion of weakly continuous multifunctions. Popa and Noiri [13] introduced a class of multifunctions called weakly α-continuous multifunctions. Popa and Noiri [14] investigated some characterizations of weakly β-continuous multifunctions. Laprom et al. [10] introduced and investigated the concept of almost $\beta(\tau_1, \tau_2)$-continuous multifunctions. Viriyapong and Boonpok [19] introduced and studied the notion weakly $(\tau_1, \tau_2)\alpha$-continuous multifunctions. Furthermore, several characterizations of weakly $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions and almost weakly (τ_1, τ_2)-continuous multifunctions were presented in [3] and [2], respectively. In this paper, we investigate the relationships between (τ_1, τ_2)-continuous multifunctions and weakly (τ_1, τ_2)-continuous multifunctions.

2 Preliminaries

Throughout this paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$*-closed* [4] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$*-open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$*-closure* [4] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$*-interior* [4] of A and is denoted by $\tau_1\tau_2$-Int(A).

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$*-paracompact* [4] if every cover of A by $\tau_1\tau_2$-open sets of X is refined by a cover of A which consists of $\tau_1\tau_2$-open sets of X and is $\tau_1\tau_2$-locally finite in X. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$*-regular* [4] if for each $x \in A$ and each $\tau_1\tau_2$-open set U of X containing x, there exists a $\tau_1\tau_2$-open set V of X such that $x \in V \subseteq \tau_1\tau_2\text{-Cl}(V) \subseteq U$.

Lemma 2.1. [4] *If A is a $\tau_1\tau_2$-regular $\tau_1\tau_2$-paracompact set of a bitopological space (X, τ_1, τ_2) and U is a $\tau_1\tau_2$-open neighbourhood of A, then there exists a $\tau_1\tau_2$-open set V of X such that $A \subseteq V \subseteq \tau_1\tau_2$-Cl$(V) \subseteq U$.*

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [1] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *upper* (τ_1, τ_2)-*continuous* [16] (resp. *upper almost* (τ_1, τ_2)-*continuous* [9],*upper weakly* (τ_1, τ_2)-*continuous* [18]) at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq V$ (resp. $F(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$, $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$). A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *upper* (τ_1, τ_2)-*continuous* (resp. *upper almost* (τ_1, τ_2)-*continuous*, *upper weakly* (τ_1, τ_2)-*continuous*) if F has this property at each point of X. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *lower* (τ_1, τ_2)-*continuous* [16] (resp. *lower almost* (τ_1, τ_2)-*continuous* [9],*lower weakly* (τ_1, τ_2)-*continuous* [18]) at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ (resp. $F(z) \cap \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) \neq \emptyset$, $F(z) \cap \sigma_1\sigma_2\text{-Cl}(V) \neq \emptyset$) for each $z \in U$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *lower* (τ_1, τ_2)-*continuous* (resp. *lower almost* (τ_1, τ_2)-*continuous*, *lower weakly* (τ_1, τ_2)-*continuous*) if F has this property at each point of X.

3 (τ_1, τ_2)-continuity and weak (τ_1, τ_2)-continuity

In this section, we investigate the relationships between (τ_1, τ_2)-continuous multifunctions and weakly (τ_1, τ_2)-continuous multifunctions.

Theorem 3.1. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ such that $F(x)$ is a $\tau_1\tau_2$-regular $\tau_1\tau_2$-paracompact set of Y for each point $x \in X$, the following properties are equivalent:*

(1) F is upper (τ_1, τ_2)-continuous;

(2) F is upper almost (τ_1, τ_2)-continuous;

(3) F is upper weakly (τ_1, τ_2)-continuous.

Proof. We show only the implication (3) $\Rightarrow$ (1) since the others are obvious. Suppose that F is upper weakly (τ_1, τ_2)-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. Since $F(x)$ is $\tau_1\tau_2$-regular $\tau_1\tau_2$-paracompact, by Lemma 2.1 there exists a $\sigma_1\sigma_2$-open set W

of Y such that $F(x) \subseteq W \subseteq \sigma_1\sigma_2$-Cl$(W) \subseteq V$. Since F is upper weakly (τ_1, τ_2)-continuous, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2$-Cl$(W)$; hence $F(U) \subseteq V$. This shows that F is upper (τ_1, τ_2)-continuous. □

Recall that a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*compact* [4] if every cover of X by $\tau_1\tau_2$-open sets of X has a finite subcover.

Definition 3.2. [6] *A bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-regular if for each $\tau_1\tau_2$-closed set F and each $x \in X - F$, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $x \in U$ and $F \subseteq V$.*

Corollary 3.3. *Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-compact for each point $x \in X$ and (Y, σ_1, σ_2) is (σ_1, σ_2)-regular. Then, the following properties are equivalent:*

(1) F is upper (τ_1, τ_2)-continuous;

(2) F is upper almost (τ_1, τ_2)-continuous;

(3) F is upper weakly (τ_1, τ_2)-continuous.

Lemma 3.4. *If A is a $\tau_1\tau_2$-regular set of a bitopological space (X, τ_1, τ_2), then for each $\tau_1\tau_2$-open set G which intersect A, there exists a $\tau_1\tau_2$-open set W such that $A \cap W \neq \emptyset$ and $\tau_1\tau_2$-Cl$(W) \subseteq G$.*

Theorem 3.5. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ such that $F(x)$ is a $\tau_1\tau_2$-regular set of Y for each point $x \in X$, the following properties are equivalent:*

(1) F is lower (τ_1, τ_2)-continuous;

(2) F is lower almost (τ_1, τ_2)-continuous;

(3) F is lower weakly (τ_1, τ_2)-continuous.

Proof. We show only the implication (3) ⇒ (1) since the others are obvious. Suppose that F is lower weakly (τ_1, τ_2)-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \cap V \neq \emptyset$. Since $F(x)$ is $\sigma_1\sigma_2$-regular, by Lemma 3.4 there exists a $\sigma_1\sigma_2$-open set W of Y such that $F(x) \cap W \neq \emptyset$ and $\sigma_1\sigma_2$-Cl$(W) \subseteq V$. Since F is lower weakly (τ_1, τ_2)-continuous, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(z) \cap \sigma_1\sigma_2$-Cl$(W) \neq \emptyset$; hence $F(z) \cap V \neq \emptyset$ for each $z \in U$. This shows that F is lower (τ_1, τ_2)-continuous. □

Definition 3.6. [7] *A bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-normal if for each pair of disjoint $\tau_1\tau_2$-closed sets F and F', there exist disjoint $\tau_1\tau_2$-open sets U and V such that $F \subseteq U$ and $F' \subseteq V$.*

Definition 3.7. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be (τ_1, τ_2)-closed if $F(K)$ is $\sigma_1\sigma_2$-closed in Y for every $\tau_1\tau_2$-closed set K of X.*

Theorem 3.8. *If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a closed valued multifunction and (Y, σ_1, σ_2) is a (σ_1, σ_2)-normal space, then the following properties are equivalent:*

(1) F is upper (τ_1, τ_2)-continuous;

(2) F is upper almost (τ_1, τ_2)-continuous;

(3) F is upper weakly (τ_1, τ_2)-continuous.

Proof. As in Theorem 3.1, we prove only the implication (3) $\Rightarrow$ (1). Suppose that F is upper weakly (τ_1, τ_2)-continuous. Let $x \in X$ and G be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Since $F(x)$ is $\sigma_1\sigma_2$-closed in Y, by the (σ_1, σ_2)-normality of Y there exists a $\tau_1\tau_2$-open set V of X such that $F(x) \subseteq V \subseteq \sigma_1\sigma_2\text{-Cl}(V) \subseteq G$. Since F is upper weakly (τ_1, τ_2)-continuous, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V) \subseteq G$. This shows that F is upper (τ_1, τ_2)-continuous. ☐

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