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(τ_1, τ_2)-continuous multifunctions and $\tau_1\tau_2$-δ-open sets

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Abstract

In this paper, we investigate some characterizations of upper and lower (τ_1, τ_2)-continuous multifunctions by utilizing the notions of $\tau_1\tau_2$-δ-open sets and $\tau_1\tau_2$-δ-closed sets.

1 Introduction

Topology is concerned with all questions directly or indirectly related to continuity. Semi-open sets, preopen sets, α-open sets, β-open sets and δ-open sets play important roles in the researches of generalizations of continuity. Using these sets, many authors introduced and investigated various types of weak forms of continuity for functions and multifunctions. In [4], the

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authors studied some properties of (Λ, sp)-open sets. Viriyapong and Boonpok [19] investigated some characterizations of (Λ, sp)-continuous functions by utilizing (Λ, sp)-open sets and (Λ, sp)-closed sets. Moreover, some characterizations of strongly $\theta(\Lambda, p)$-continuous functions were studied in [18]. Popa and Noiri [15] obtained some characterizations of upper and lower α-continuous multifunctions. Furthermore, Popa and Noiri [14] introduced and studied the notions of upper and lower β-continuous multifunctions. Noiri and Popa [11] investigated the concepts upper and lower M-continuous multifunctions as multifunctions defined between sets satisfying certain minimal conditions. Popa and Noiri [13] introduced and studied the notion of m-continuous multifunctions. Park et al. [12] introduced and investigated δ-precontinuous multifunctions as a generalization of precontinuous multifunctions due to Popa [16]. Laprom et al. [10] introduced and studied the notions of upper and lower $\beta(\tau_1, \tau_2)$-continuous multifunctions. Viriyapong and Boonpok [20] introduced and investigated the concepts of upper and lower $(\tau_1, \tau_2)\alpha$-continuous multifunctions. Additionally, several characterizations of $(\tau_1, \tau_2)\delta$-semicontinuous mutifunctions and almost weakly (τ_1, τ_2)-continuous multifunctions, $\star$-continuous multifunctions and $\beta(\star)$-continuous multifunctions were presented in [7], [5], [8] and [6], respectively. In this paper, we investigate some characterizations of upper and lower (τ_1, τ_2)-continuous multifunctions by utilizing the notions of $\tau_1\tau_2$-δ-closed sets and $\tau_1\tau_2$-δ-open sets.

2 Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [9] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [9] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [9] of A and is denoted by $\tau_1\tau_2$-Int(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [20] (resp. $(\tau_1, \tau_2)s$-*open* [7]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)))$. The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open) set is said to be $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*). Let A be a subset of a bitopological space

(X, τ_1, τ_2). The intersection of all $(\tau_1, \tau_2)s$-closed sets of X containing A is called the $(\tau_1, \tau_2)s$-*closure* [7] of A and is denoted by (τ_1, τ_2)-sCl(A). The union of all $(\tau_1, \tau_2)s$-open sets of X contained in A is called the $(\tau_1, \tau_2)s$-*interior* [7] of A and is denoted by (τ_1, τ_2)-sInt(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-δ-*open* [3] if A is the union of $(\tau_1, \tau_2)r$-open sets of X. The complement of a $\tau_1\tau_2$-δ-open set is called $\tau_1\tau_2$-δ-*closed* [3]. The union of all $\tau_1\tau_2$-δ-open sets of X contained in A is called the $\tau_1\tau_2$-δ-*interior* [3] of A and is denoted by $\tau_1\tau_2$-δ-Int(A). The intersection of all $\tau_1\tau_2$-δ-closed sets of X containing A is called the $\tau_1\tau_2$-δ-*closure* [3] of A and is denoted by $\tau_1\tau_2$-δ-Cl(A).

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [1] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3 Characterizations of upper and lower (τ_1, τ_2)-continuous multifunctions

In this section, we investigate some characterizations of upper and lower (τ_1, τ_2)-continuous multifunctions.

Definition 3.1. [17] *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper (τ_1, τ_2)-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq V$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower (τ_1, τ_2)-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for each $z \in U$.*

Lemma 3.2. [17] *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper (τ_1, τ_2)-continuous;

(2) $F^+(V)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $F^-(K)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2$-*Cl*$(F^-(B)) \subseteq F^-(\sigma_1\sigma_2$-*Cl*$(B))$ *for every subset* B *of* Y*;*

(5) $F^+(\sigma_1\sigma_2$-*Int*$(B)) \subseteq \tau_1\tau_2$-*Int*$(F^+(B))$ *for every subset* B *of* Y*.*

Definition 3.3. *A bitopological space* (X, τ_1, τ_2) *is said to be* $(\tau_1, \tau_2)s$*-regular if for each* $(\tau_1, \tau_2)s$*-closed set* F *and each* $x \notin F$*, there exist disjoint* $(\tau_1, \tau_2)s$*-open sets* U *and* V *such that* $x \in U$ *and* $F \subseteq V$*.*

Lemma 3.4. *A bitopological space* (X, τ_1, τ_2) *is* $(\tau_1, \tau_2)s$*-regular if and only if for each* $x \in X$ *and each* $(\tau_1, \tau_2)s$*-open set* U *containing* x*, there exists a* $(\tau_1, \tau_2)s$*-open set* V *such that* $x \in V \subseteq (\tau_1, \tau_2)$*-sCl*$(V) \subseteq U$*.*

Lemma 3.5. *Let* (X, τ_1, τ_2) *be a* $(\tau_1, \tau_2)s$*-regular space. Then, the following properties hold:*

(1) $\tau_1\tau_2$-*Cl*$(A) = \tau_1\tau_2$-δ-*Cl*(A) *for every subset* A *of* X*.*

(2) Every $\tau_1\tau_2$*-open set is* $\tau_1\tau_2$-δ*-open.*

Theorem 3.6. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is upper* (τ_1, τ_2)*-continuous;*

(2) $F^-(\sigma_1\sigma_2$-δ-*Cl*$(B))$ *is* $\tau_1\tau_2$*-closed in* X *for every subset* B *of* Y*;*

(3) $F^-(K)$ *is* $\tau_1\tau_2$*-closed in* X *for every* $\sigma_1\sigma_2$-δ*-closed set* K *of* Y*;*

(4) $F^+(V)$ *is* $\tau_1\tau_2$*-open in* X *for every* $\sigma_1\sigma_2$-δ*-open set* V *of* Y*.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. By Lemma 3.5, $\sigma_1\sigma_2$-δ-Cl(B) is $\sigma_1\sigma_2$-closed in Y. Since F is upper (τ_1, τ_2)-continuous, by Lemma 3.2 $F^-(\sigma_1\sigma_2$-δ-Cl$(B))$ is $\tau_1\tau_2$-closed in X.

(2) $\Rightarrow$ (3): Let K be any $\sigma_1\sigma_2$-δ-closed set of Y. Then, $\sigma_1\sigma_2$-δ-Cl$(K) = K$ and by (2), we have $F^-(K)$ is $\tau_1\tau_2$-closed in X.

(3) $\Rightarrow$ (4): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for any subset B of Y.

(4) $\Rightarrow$ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Since (Y, σ_1, σ_2) is $(\sigma_1, \sigma_2)s$-regular, we have V is $\sigma_1\sigma_2$-δ-open in Y and by (4), $F^+(V)$ is $\tau_1\tau_2$-open in X. Thus, F is upper (τ_1, τ_2)-continuous by Lemma 3.2. □

Theorem 3.7. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F is lower (τ_1, τ_2)-continuous;

(2) $F^+(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))$ is $\tau_1\tau_2$-closed in X for every subset B of Y;

(3) $F^+(K)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-δ-closed set K of Y;

(4) $F^-(V)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-δ-open set V of Y.

Proof. The proof is similar to that of Theorem 3.6. □

Definition 3.8. [2] *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called (τ_1, τ_2)-continuous if f has this property at each point of X.*

Corollary 3.9. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is (τ_1, τ_2)-continuous;

(2) $f^{-1}(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))$ is $\tau_1\tau_2$-closed in X for every subset B of Y;

(3) $f^{-1}(K)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-δ-closed set K of Y;

(4) $f^{-1}(V)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-δ-open set V of Y.

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