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On almost (τ_1, τ_2)-normal spaces

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Abstract

In this paper, we introduce the notion of almost (τ_1, τ_2)-normal spaces. We also investigate some properties of almost (τ_1, τ_2)-normal spaces.

1 Introduction

It is well known that various types of separation axioms play a significant role in the theory of classical point set topology. In literature, separation axioms have been studied by many mathematicians. In 1971, Viglino [17] introduced the notion of seminormal spaces. Singal and Arya [14] introduced the class of almost normal spaces and proved that a space is normal

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if and only if it is both a seminormal space and an almost normal space. In 1995, Paul and Bhattacharyya [12] introduced and studied the notion of p-normal spaces. Singal and Singal [13] introduced a new weak form of normal spaces called mildly normal spaces. Navalagi [10] have continued the study of further properties of p-normal spaces and also introduced and studied mildly p-normal (resp. almost p-normal) spaces which are generalizations of both mildly normal (resp. almost normal) spaces and p-normal spaces. Park [11] obtained some characterizations of almost p-normal spaces and mildly p-normal spaces. On the other hand, the notions of δp-normal spaces, almost δp-normal spaces and mildly δp-normal spaces were introduced by Ekici and Noiri [7]. Buadong et al. [5] introduced and discussed new separation axioms in generalized topology and minimal structure spaces. Srisarakham and Boonpok [15] studied some weak separation axioms by utilizing $\delta p(\Lambda, s)\mathscr{D}$-sets. In [2], the present authors studied some properties of (Λ, sp)-open sets. Boonpok and Viriyapong [1] introduced and investigated some weak separation axioms by utilizing the notions of (Λ, sp)-open sets and the (Λ, sp)-closure operator. Moreover, some properties of $\mu_{(m,n)}$-normal spaces, (Λ, p)-normal spaces and $S\Lambda_s$-normal spaces were presented in [16], [9] and [8], respectively. In this paper, we introduce the notion of almost (τ_1, τ_2)-normal spaces. Furthermore, we investigate some properties of almost (τ_1, τ_2)-normal spaces.

2 Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [4] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [4] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [4] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 2.1. [4] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) *If $A \subseteq B$, then $\tau_1\tau_2$-Cl$(A) \subseteq \tau_1\tau_2$-Cl$(B)$.*

(3) *$\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed.*

(4) *A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl(A).*

(5) *$\tau_1\tau_2$-Cl$(X - A) = X - \tau_1\tau_2$-Int$(A)$.*

A subset A of a bitopological space (X, τ_1, τ_2) is said to be *$(\tau_1, \tau_2)r$-open* [19] (resp. *$(\tau_1, \tau_2)s$-open* [3], *$(\tau_1, \tau_2)p$-open* [3], *$(\tau_1, \tau_2)\beta$-open* [3]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A)))$). A subset A of a bitopological space (X, τ_1, τ_2) is said to be *generalized (τ_1, τ_2)-closed* (briefly, g-(τ_1, τ_2)-closed) [18] if $\tau_1\tau_2$-Cl$(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_1\tau_2$-open. A subset A is called *g-(τ_1, τ_2)-open* [18] if $X - A$ is g-(τ_1, τ_2)-closed.

Lemma 2.2. [18] *A subset A of a bitopological space (X, τ_1, τ_2) is g-(τ_1, τ_2)-open if and only if $F \subseteq \tau_1\tau_2$-Int(A) whenever $F \subseteq A$ and F is $\tau_1\tau_2$-closed.*

A subset A of a bitopological space (X, τ_1, τ_2) is said to be *regular generalized (τ_1, τ_2)-closed* (briefly, rg-(τ_1, τ_2)-closed) [6] if $\tau_1\tau_2$-Cl$(A) \subseteq U$ whenever $A \subseteq U$ and U is $(\tau_1, \tau_2)r$-open. A subset A is called *rg-(τ_1, τ_2)-open* [6] if $X - A$ is rg-(τ_1, τ_2)-closed.

Lemma 2.3. [6] *A subset A of a bitopological space (X, τ_1, τ_2) is rg-(τ_1, τ_2)-open if and only if $F \subseteq \tau_1\tau_2$-Int(A) whenever $F \subseteq A$ and F is $\tau_1\tau_2$-closed.*

3 Almost (τ_1, τ_2)-normal spaces

In this section, we introduce the notion of almost (τ_1, τ_2)-normal spaces. Moreover, we investigate some properties of almost (τ_1, τ_2)-normal spaces.

Definition 3.1. *A bitopological space (X, τ_1, τ_2) is said to be almost (τ_1, τ_2)-normal if for each $\tau_1\tau_2$-closed set A and each $(\tau_1, \tau_2)r$-closed set B such that $A \cap B = \emptyset$, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*

Theorem 3.2. *For a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) *(X, τ_1, τ_2) is almost (τ_1, τ_2)-normal;*

(2) for each $\tau_1\tau_2$-closed set A and each $(\tau_1, \tau_2)r$-closed set B such that

$$A \cap B = \emptyset,$$

there exist disjoint g-(τ_1, τ_2)-open sets U and V such that $A \subseteq U$ and $B \subseteq V$;

(3) for each $\tau_1\tau_2$-closed set A and each $(\tau_1, \tau_2)r$-closed set B such that

$$A \cap B = \emptyset,$$

there exist disjoint rg-(τ_1, τ_2)-open sets U and V such that $A \subseteq U$ and $B \subseteq V$;

(4) for each $\tau_1\tau_2$-closed set F and each $(\tau_1, \tau_2)r$-open set U containing F, there exists a rg-(τ_1, τ_2)-open set V such that $F \subseteq V \subseteq \tau_1\tau_2$-Cl$(V) \subseteq U$;

(5) for each rg-(τ_1, τ_2)-closed set F and each $(\tau_1, \tau_2)r$-open set U containing F, there exists a $\tau_1\tau_2$-open set V such that $\tau_1\tau_2$-Cl$(F) \subseteq V \subseteq \tau_1\tau_2$-Cl$(V) \subseteq U$;

(6) for each rg-(τ_1, τ_2)-closed set F and each $(\tau_1, \tau_2)r$-open set U containing F, there exists a g-(τ_1, τ_2)-open set V such that $\tau_1\tau_2$-Cl$(F) \subseteq V \subseteq \tau_1\tau_2$-Cl$(V) \subseteq U$;

(7) for each rg-(τ_1, τ_2)-closed set F and each $(\tau_1, \tau_2)r$-open set U containing F, there exists a rg-(τ_1, τ_2)-open set V such that $\tau_1\tau_2$-Cl$(F) \subseteq V \subseteq \tau_1\tau_2$-Cl$(V) \subseteq U$.

Proof. It is obvious that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (4).

(3) $\Rightarrow$ (4): Let F be a $\tau_1\tau_2$-closed set and U be a $(\tau_1, \tau_2)r$-open set containing F. Then, $F \cap (X - U) = \emptyset$ and by (3), there exist disjoint rg-(τ_1, τ_2)-open sets V and W such that $F \subseteq V$ and $X - U \subseteq W$. By Lemma 2.3, we have $X - U \subseteq \tau_1\tau_2\text{-Int}(W)$ and $V \cap \tau_1\tau_2\text{-Int}(W) = \emptyset$. Thus,

$$\tau_1\tau_2\text{-Cl}(V) \cap \tau_1\tau_2\text{-Int}(W) = \emptyset$$

and hence $F \subseteq V \subseteq \tau_1\tau_2\text{-Cl}(V) \subseteq X - \tau_1\tau_2\text{-Int}(W) \subseteq U$.

(4) $\Rightarrow$ (1): Let A be any $\tau_1\tau_2$-closed set and B be any $(\tau_1, \tau_2)r$-closed set such that $A \cap B = \emptyset$. Then, $X - B$ is a $(\tau_1, \tau_2)r$-open set containing A and there exists a rg-(τ_1, τ_2)-open sets U such that $A \subseteq U \subseteq \tau_1\tau_2\text{-Cl}(U) \subseteq X - B$. Put $V = \tau_1\tau_2\text{-Int}(U)$ and $W = X - \tau_1\tau_2\text{-Cl}(U)$. Then, V and W are disjoint

$\tau_1\tau_2$-open sets such that $A \subseteq V$ and $B \subseteq W$. This shows that (X, τ_1, τ_2) is almost (τ_1, τ_2)-normal.

(1) $\Rightarrow$ (5): Let F be a rg-(τ_1, τ_2)-closed set and U be a $(\tau_1, \tau_2)r$-open set containing F. Then, $\tau_1\tau_2$-Cl$(F) \subseteq U$ and hence $\tau_1\tau_2$-Cl$(F) \cap (X - U) = \emptyset$. By (1), there exist disjoint $\tau_1\tau_2$-open sets V and W such that $\tau_1\tau_2$-Cl$(F) \subseteq V$ and $X - U \subseteq W$. Thus, $\tau_1\tau_2$-Cl$(F) \subseteq V \subseteq \tau_1\tau_2$-Cl$(V) \subseteq X - W \subseteq U$. □

Definition 3.3. *A bitopological space (X, τ_1, τ_2) is said to be mildly (τ_1, τ_2)-normal if for every pair of disjoint $(\tau_1, \tau_2)r$-closed sets A and B, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*

Theorem 3.4. *For a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) (X, τ_1, τ_2) is mildly (τ_1, τ_2)-normal;

(2) for any disjoint $(\tau_1, \tau_2)r$-closed sets A and B, there exist disjoint g-(τ_1, τ_2)-open sets U and V such that $A \subseteq U$ and $B \subseteq V$;

(3) for any disjoint $(\tau_1, \tau_2)r$-closed sets A and B, there exist disjoint rg-(τ_1, τ_2)-open sets U and V such that $A \subseteq U$ and $B \subseteq V$;

(4) for each $(\tau_1, \tau_2)r$-closed set F and each $(\tau_1, \tau_2)r$-open set U containing F, there exists a g-(τ_1, τ_2)-open set V such that $F \subseteq V \subseteq \tau_1\tau_2$-Cl$(V) \subseteq U$;

(5) for each $(\tau_1, \tau_2)r$-closed set F and each $(\tau_1, \tau_2)r$-open set U containing F, there exists a rg-(τ_1, τ_2)-open set V such that $F \subseteq V \subseteq \tau_1\tau_2$-Cl$(V) \subseteq U$.

Proof. The proof is similar to that of Theorem 3.2. □

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