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On almost (τ_1, τ_2)-regular spaces

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Abstract

In this paper, we introduce the notion of almost (τ_1, τ_2)-regular spaces. Furthermore, we investigate some characterizations of almost (τ_1, τ_2)-regular spaces.

1 Introduction

Separation axioms are one among the most common, important and interesting ideas in topology. Some separation axioms were introduced using generalized open sets. In 1969, Sinnal and Arya [13] defined a new separation axiom called almost regularity which is weaker than regularity. In 1983, El-Deeb et al. [7] introduced and studied the notion of p-regular spaces by using preopen

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sets. In 1990, Malghan and Navalagi [9] introduced and investigated the concept of almost p-regular spaces as a generalization of p-regularity. In 1998, Noiri [10] defined a new class of sets called rgp-closed sets and investigated some characterizations of almost p-regular spaces by utilizing rgp-closed sets. Buadong et al. [5] introduced and studied new separation axioms in generalized topology and minimal structure spaces. Srisarakham and Boonpok [14] investigated some weak separation axioms by utilizing $\delta p(\Lambda, s)\mathscr{D}$-sets. In [2], the present authors studied some properties of (Λ, sp)-open sets. Boonpok and Viriyapong [1] introduced and studied some weak separation axioms by utilizing the notions of (Λ, sp)-open sets and the (Λ, sp)-closure operator. In 2012, Torton et al. [15] introduced and investigated the notion of $\mu_{(m,n)}$-regular spaces. Furthermore, several characterizations of (Λ, p)-regular spaces and $S\Lambda_s$-regular spaces were presented in [11] and [8], respectively. Roy [12] obtained some characterizations of almost μ-regular spaces. In this paper, we introduce the notion of almost (τ_1, τ_2)-regular spaces. Moreover, we discuss some characterizations of almost (τ_1, τ_2)-regular spaces.

2 Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [4] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [4] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [4] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 2.1. [4] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [17] (resp. $(\tau_1, \tau_2)s$-*open* [3], $(\tau_1, \tau_2)p$-*open* [3], $(\tau_1, \tau_2)\beta$-*open* [3]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$-*open* [16] if $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is called $\alpha(\tau_1, \tau_2)$-*closed*.

Definition 2.2. [6] *A subset A of a bitopological space (X, τ_1, τ_2) is said to be regular generalized (τ_1, τ_2)-closed (briefly, rg-(τ_1, τ_2)-closed) if $\tau_1\tau_2\text{-}Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $(\tau_1, \tau_2)r$-open.*

Definition 2.3. [6] *A subset A of a bitopological space (X, τ_1, τ_2) is said to be regular generalized (τ_1, τ_2)-open (briefly, rg-(τ_1, τ_2)-open) if $X - A$ is regular generalized (τ_1, τ_2)-closed.*

Lemma 2.4. [6] *A subset A of a bitopological space (X, τ_1, τ_2) is rg-(τ_1, τ_2)-open if and only if $F \subseteq \tau_1\tau_2\text{-}Int(A)$ whenever $F \subseteq A$ and F is $(\tau_1, \tau_2)r$-closed.*

3 Almost (τ_1, τ_2)-regular spaces

In this paper, we introduce the notion of almost (τ_1, τ_2)-regular spaces. We also investigate some characterizations of almost (τ_1, τ_2)-regular spaces.

Definition 3.1. *A bitopological space (X, τ_1, τ_2) is said to be almost (τ_1, τ_2)-regular if for each $(\tau_1, \tau_2)r$-closed set F and each $x \notin F$, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $x \in U$ and $F \subseteq V$.*

Theorem 3.2. *For a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) (X, τ_1, τ_2) is almost (τ_1, τ_2)-regular;

(2) for each $x \in X$ and each $(\tau_1, \tau_2)r$-open set U with $x \in U$, there exists a $\tau_1\tau_2$-open set V such that $x \in V \subseteq \tau_1\tau_2\text{-}Cl(V) \subseteq U$;

(3) for each $(\tau_1, \tau_2)r$-closed set F of X,

$$\cap\{\tau_1\tau_2\text{-}Cl(V) \mid F \subseteq V \text{ and } V \text{ is } \tau_1\tau_2\text{-open}\} = F;$$

(4) *for each subset A of X and each $(\tau_1, \tau_2)r$-open set U of X such that $A \cap U \neq \emptyset$, there exists a $\tau_1\tau_2$-open set V such that $A \cap V \neq \emptyset$ and $\tau_1\tau_2$-Cl$(V) \subseteq U$;*

(5) *for each nonempty subset A of X and each $(\tau_1, \tau_2)r$-closed set F such that $A \cap F = \emptyset$, there exist $\tau_1\tau_2$-open sets U and V such that $A \cap U \neq \emptyset$, $F \subseteq V$ and $U \cap V = \emptyset$;*

(6) *for each $(\tau_1, \tau_2)r$-closed set F and $x \notin F$, there exist a $\tau_1\tau_2$-open set U and a rg-(τ_1, τ_2)-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$;*

(7) *for each subset A of X and each $(\tau_1, \tau_2)r$-closed set F such that $A \cap F = \emptyset$, there exist a $\tau_1\tau_2$-open set U and a rg-(τ_1, τ_2)-open set V such that $A \cap U \neq \emptyset$, $F \subseteq V$ and $U \cap V = \emptyset$.*

Proof. (1) ⇒ (2): Let U be a $(\tau_1, \tau_2)r$-open set with $x \in U$. Then, $X - U$ is $(\tau_1, \tau_2)r$-closed and $x \notin X - U$. Then by (1), there exist disjoint $\tau_1\tau_2$-open sets V and W such that $x \in V$ and $X - U \subseteq W$. Thus,

$$x \in V \subseteq \tau_1\tau_2\text{-Cl}(V) \subseteq \tau_1\tau_2\text{-Cl}(X - W) = X - W \subseteq U.$$

(2) ⇒ (3): Let F be a $(\tau_1, \tau_2)r$-closed set and $x \notin X - F$. By (2), there exists a $\tau_1\tau_2$-open set U such that $x \in U \subseteq \tau_1\tau_2$-Cl$(U) \subseteq X - F$. Therefore, $F \subseteq X - \tau_1\tau_2$-Cl$(U) = V$. Since V is $\tau_1\tau_2$-open and $U \cap V = \emptyset$. Thus, $x \notin \tau_1\tau_2$-Cl(V) and hence $F \supseteq \cap\{\tau_1\tau_2\text{-Cl}(V) \mid F \subseteq V$ and V is $\tau_1\tau_2$-open$\}$.

(3) ⇒ (4): Let A be a subset of X and U be a $(\tau_1, \tau_2)r$-open set such that $A \cap U \neq \emptyset$. Let $x \in A \cap U$. Then, we have $x \notin X - U$. Thus by (3), there exists a $\tau_1\tau_2$-open set W such that $X - U \subseteq W$ and $x \notin \tau_1\tau_2$-Cl(W). Put $V = X - \tau_1\tau_2$-Cl(W) which is a $\tau_1\tau_2$-open set containing x and hence $A \cap V \neq \emptyset$. Now, $V \subseteq X - W$ and so $\tau_1\tau_2$-Cl$(V) \subseteq X - W \subseteq U$.

(4) ⇒ (5): Let A be a nonempty subset of X and F be a $(\tau_1, \tau_2)r$-closed set such that $A \cap F = \emptyset$. Then, $X - F$ is $(\tau_1, \tau_2)r$-open and $A \cap (X - F) \neq \emptyset$. By (4), there exists a $\tau_1\tau_2$-open set V such that $A \cap V \neq \emptyset$ and

$$\tau_1\tau_2\text{-Cl}(V) \subseteq X - F.$$

If we put $W = X - \tau_1\tau_2$-Cl(V), then $F \subseteq W$ and $W \cap V = \emptyset$.

(5) ⇒ (1): Let F be a $(\tau_1, \tau_2)r$-closed set not containing x. Then, we have $F \cap \{x\} = \emptyset$. Thus by (5), there exist $\tau_1\tau_2$-open sets V and W such that $x \in V$, $F \subseteq W$ and $V \cap W = \emptyset$.

(1) ⇒ (6): The proof is obvious.

(6) $\Rightarrow$ (7): Let A be a subset of X and F be a $(\tau_1, \tau_2)r$-closed set such that $A \cap F = \emptyset$. Then, for each $x \in A$, $x \notin F$. By (6), there exist a $\tau_1\tau_2$-open set U and a rg-(τ_1, τ_2)-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$. Thus, $A \cap U \neq \emptyset$, $F \subseteq V$ and $U \cap V = \emptyset$.

(7) $\Rightarrow$ (1): Let F be a $(\tau_1, \tau_2)r$-closed set such that $x \notin F$. Since $\{x\} \cap F = \emptyset$, by (7) there exist a $\tau_1\tau_2$-open set U and a rg-(τ_1, τ_2)-open set W such that $x \in U$, $F \subseteq W$ and $U \cap W = \emptyset$. Since W is rg-(τ_1, τ_2)-open, by Lemma 2.4, $F \subseteq \tau_1\tau_2$-Cl$(W) = V$ and $U \cap V = \emptyset$. This shows that (X, τ_1, τ_2) is almost (τ_1, τ_2)-regular. □

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